**TEORIA DA LIBRO LOSS PANJER**



9

AGGREGATE LOSS MODELS

9.1 INTRODUCTION

An insurance enterprise exists because of its ability to pool risks. By insuring many people the individual risks are combined into an aggregate risk that is manageable and can be priced at a level that will attract customers. Consider the following simple example.

EXAMPLE 9.1

An insurable event has a 10% probability of occurring and when it occurs results in a loss of 5,000. Market research has indicated that consumers will pay at most 550 to purchase insurance against this event. How many policies must a company sell in order to have a 95% chance of making money (ignoring expenses)?

Let be the number of policies sold. A reasonable model for the number of claims, is a binomial distribution with = and = 01 and the total

Loss Models: From Data to Decisions, 3rd. ed. By Stuart A. Klugman, Harry H. Panjer, 209 Gordon E. Willmot Copyright °c 2008 John Wiley & Sons, Inc.

     

   

210 AGGREGATE LOSS MODELS paid will be 5000. To achieve the desired outcome,

095 Pr(5000 550) = Pr( 011)

Ã 01101! = Pr p01(09)

 

where the approximation uses the Central Limit Theorem. With the normal

distribution

which gives the answer = 3,45744, and so at least 3,458 policies must be

011 01 p01(09) =196

¤

 

sold.

The goal of this chapter is to build a model for the total payments by an insurance system (which may be the entire company, a line of business, or those covered by a group insurance contract). The building blocks are random variables that describe the number of claims and the amounts of those claims, subjects covered in the previous chapters.

There are two ways to build a model for the amount paid on all claims occurring in a xed time period on a dened set of insurance contracts. The rst is to record the payments as they are made and then add them up. In that case we can represent the aggregate losses as a sum, , of a random number, , of individual payment amounts (1 2 ). Hence,

=1+2+···+ =012 (9.1) where = 0 when = 0.

Denition 9.1 The collective risk model has the representation in (9.1) with the s being independent and identically distributed (i.i.d.) random variables, unless otherwise specied. More formally, the independence assumptions are

1. Conditional on = , the random variables 1 2 are i.i.d. random variables.

2. Conditional on = , the common distribution of the random variables 12 does not depend on .

3. The distribution of does not depend in any way on the values of 1 2 .

The second model, the one used in Example 9.1, assigns a random variable to each contract.

Denition 9.2 The individual risk model represents the aggregate loss as a sum, = 1 + · · · + , of a xed number, , of insurance contracts. The loss amounts for the contracts are (12··· ), where the s are assumed to be independent but are not assumed to be identically distributed. The distribution

   

   

INTRODUCTION 211 of the s usually has a probability mass at zero, corresponding to the probability

of no loss or payment.

The individual risk model is used to add together the losses or payments from a xed number of insurance contracts or sets of insurance contracts. It is used in modeling the losses of a group life or health insurance policy that covers a group of employees. Each employee can have dierent coverage (life insurance benet as a multiple of salary) and dierent levels of loss probabilities (dierent ages and health status).

In the special case where the s are identically distributed, the individual risk model becomes the special case of the collective risk model, with the distribution of being the degenerate distribution with all of the probability at = ; that is, Pr( = ) = 1.

The distribution of in (9.1) is obtained from the distribution of and the distribution of the s. Using this approach, the frequency and the severity of claims are modeled separately. The information about these distributions is used to obtain information about . An alternative to this approach is to simply gather information about (e.g., total losses each month for a period of months) and to use some model from the earlier chapters to model the distribution of . Modeling the distribution of and the distribution of the s separately has seven distinct advantages:

* The expected number of claims changes as the number of insured policies changes. Growth in the volume of business needs to be accounted for in forecasting the number of claims in future years based on past years’ data.
* The eects of general economic ination and additional claims ination are reected in the losses incurred by insured parties and the claims paid by insur- ance companies. Such eects are often masked when insurance policies have deductibles and policy limits that do not depend on ination and aggregate results are used.
* The impact of changing individual deductibles and policy limits is easily im- plemented by changing the specication of the severity distribution.
* The impact on claims frequencies of changing deductibles is better under- stood.
* Data that are heterogeneous in terms of deductibles and limits can be com- bined to obtain the hypothetical loss size distribution. This approach is useful when data from several years in which policy provisions were changing are combined.
* Models developed for noncovered losses to insureds, claim costs to insurers, and claim costs to reinsurers can be mutually consistent. This feature is useful for a direct insurer when studying the consequence of shifting losses to a reinsurer.
* The shape of the distribution of depends on the shapes of both distributions of and . The understanding of the relative shapes is useful when modify- ing policy details. For example, if the severity distribution has a much heavier

   

   

212 AGGREGATE LOSS MODELS

tail than the frequency distribution, the shape of the tail of the distribution of aggregate claims or losses will be determined by the severity distribution and will be insensitive to the choice of frequency distribution.

In summary, a more accurate and exible model can be constructed by examining frequency and severity separately.

In constructing the model (9.1) for , if represents the actual number of losses to the insured, then the s can represent (i) the losses to the insured, (ii) the claim payments of the insurer, (iii) the claim payments of a reinsurer, or (iv) the deductibles (self-insurance) paid by the insured. In each case, the interpretation of is dierent and the severity distribution can be constructed in a consistent manner.

Because the random variables , 12, and provide much of the focus for this chapter and Chapters 10 and 11, we want to be especially careful when referring to them. To that end, we refer to as the claim count random variable and refer to its distribution as the claim count distribution. The expression number of claims is also used, and, occasionally, just claims. Another term commonly used is frequency distribution. The s are the individual or single-loss random variables. The modier individual or single is dropped when the reference is clear. In Chapter 8, a distinction is made between losses and payments. Strictly speaking, the s are payments because they represent a real cash transaction. However, the term loss is more customary, and we continue with it. Another common term for the s is severity. Finally, is the aggregate loss random variable or the total loss random variable.

EXAMPLE 9.2

Describe how a collective risk model could be used for the total payments made in one year on an automobile physical damage policy with a deductible of 250.

There are two ways to do this. First, let be the number of accidents, including those that do not exceed the deductible. The individual loss vari- ables are the variables from Chapter 8. The other way is to let count the number of payments. In this case the individual loss variable is . ¤

9.1.1 Exercises

9.1 Show how the model in Example 9.1 could be written as a collective risk model. 9.2 For each of the following situations, which model (individual or collective) is

more likely to provide a better description of aggregate losses?

* (a)  A group life insurance contract where each employee has a dierent age, gender, and death benet.
* (b)  A reinsurance contract that pays when the annual total medical mal- practice costs at a certain hospital exceeds a given amount.
* (c)  A dental policy on an individual pays for at most two check-ups per year per family member. A single contract covers any size family at the same price.

    

   

9.2 MODEL CHOICES

In many cases of tting frequency or severity distributions to data, several distri- butions may be good candidates for models. However, some distributions may be preferable for a variety of practical reasons.

In general, it is useful for the severity distribution to be a scale distribution (see Denition 4.2) because the choice of currency (e.g., U.S. dollars or British pounds) should not aect the result. Also, scale families are easy to adjust for inationary eects over time (this is, in eect, a change in currency; e.g., 1994 U.S. dollars to 1995 U.S. dollars). When forecasting the costs for a future year, the anticipated rate of ination can be factored in easily by adjusting the parameters.

A similar consideration applies to frequency distributions. As a block of an insurance company’s business grows, the number of claims can be expected to grow, all other things being equal. Models that have probability generating functions of the form

(; ) = () (9.2)

for some parameter have the expected number of claims proportional to . In- creasing the volume of business by 100% results in expected claims being propor- tional to = (1 + ). This approach is discussed in Section 6.12. Because is any value satisfying 1, the distributions satisfying (9.2) should allow to take on any positive values. Such distributions can be shown to be innitely divisible (see Denition 6.17).

A related consideration, the concept of invariance over the time period of the study, also supports using frequency distributions that are innitely divisible. Ide- ally the model selected should not depend on the length of the time period used in the study of claims frequency. In particular, the expected frequency should be proportional to the length of the time period after any adjustment for growth in business. In this case, a study conducted over a period of 10 years can be used to develop claims frequency distributions for periods of one month, one year, or any other period. Furthermore, the form of the distribution for a one-year period is the same as for a one-month period with a change of parameter. The parameter corresponds to the length of a time period. For example, if = 17 in (9.2) for a one-month period, then the identical model with = 204 is an appropriate model for a one-year period.

Distributions that have a modication at zero are not of the form (9.2). However, it may still be desirable to use a zero-modied distribution if the physical situation suggests it. For example, if a certain proportion of policies never make a claim, due to duplication of coverage or other reason, it may be appropriate to use this same proportion in future periods for a policy selected at random.

9.2.1 Exercises

9.3 For pgfs satisfying (9.2), show that the mean is proportional to .

9.4 Which of the distributions in Appendix B satisfy (9.2) for any positive value of ?

MODEL CHOICES 213

   

   

214 AGGREGATE LOSS MODELS 9.3 THE COMPOUND MODEL FOR AGGREGATE CLAIMS

Let denote aggregate losses associated with a set of observed claims 1 2 · · · satisfying the independence assumptions following (9.1). The approach in this chapter involves the following three steps:

1. Develop a model for the distribution of based on data.
2. Develop a model for the common distribution of the s based on data.
3. Using these two models, carry out necessary calculations to obtain the distri- bution of .

Completion of the rst two steps follows the ideas developed elsewhere in this text. We now presume that these two models are developed and that we only need to carry out numerical work in obtaining solutions to problems associated with the distribution of . These might involve pricing a stop-loss reinsurance contract, and they require analyzing the impact of changes in deductibles, coinsurance levels, and maximum payments on individual losses.

The random sum (where has a counting distribution) has distribution function

() = PXr()

= Pr( | = ) =0

= X ()

=0

= 1 + 2 + · · · +

where () = Pr( ) is the common distribution function of the s and

= Pr( = ). The distribution of is called a compound distribution. In (9.3),

() is the “-fold convolution” of the cdf of . It can be obtained as

0()=1⁄2 0 0 1 0

and Z ()= (1)() ()for=12

(9.4)

(9.5)

The tail may then be written, for all 0, as 1 ()=X [1()]

=1

If is a continuous random variable with probability zero on nonpositive values, (9.4) reduces to

Z (1) ()= ()()for=23

0

(9.3)

   

   

THE COMPOUND MODEL FOR AGGREGATE CLAIMS 215 For = 1 this equation reduces to 1() = (). By dierentiating, the pdf is

 ()= ()()for=23

Z (1) 0

Therefore, if is continuous, then has a pdf, which, for 0, is given by ( ) = X ( ) ,

( 9 . 6 )

=1

andadiscretemasspoint,Pr(=0)=0 at=0. NotethatPr(=0)6= (0)=lim0+ ().

If has a discrete counting distribution, with probabilities at 0 1 2 , (9.4) reduces to

X ()= (1)() ()for=01 =23

The corresponding pf is

=0

X ()= (1)() ()for=01 =23

For notational purposes, let 0(0) = 1 and 0() = 0 for 6= 0. Then, in this

=0

case, has a discrete distribution with pf ()=Pr(=)=X () =01 .

Arguing as in Section 6.8, the pgf of is

E[]

* =  E[0] Pr( = 0) + X E[1+2+···+ | = ] Pr( = )  =1 X Y
* =  Pr( =0)+ E =1 =1
* =  X P r ( = ) [ ( ) ] =0
* =  E[ () ] = [ ()]

(9.7)

() =

=0

due to the independence of 1 for xed . The pgf is typically used when is discrete.

A similar relationship exists for the other generating functions. It is sometimes more convenient to use the characteristic function

() = E() = [()]

Pr( =)

(9.8)

   

   

216 AGGREGATE LOSS MODELS

which always exists. Panjer and Willmot [138] use the Laplace transform () = E() = [()]

which always exists for random variables dened on nonnegative values. With regard to the moment generating function, we have

() = [ ()]

The pgf of compound distributions is discussed in Section 6.8 where the “secondary” distribution plays the role of the claim size distribution in this chapter. In that section the claim size distribution is always discrete.

In the case where () = 1[2()] (i.e., is itself a compound distribution), () = 1{2[()]}, which in itself produces no additional diculties.

From (9.8), the moments of can be obtained in terms of the moments of and the s. The rst three moments are

E() = Var() = E{[ ()]3} =

01 = 0101 = E()E() 2 = 012 + 2(01)2 (9.9) 3 = 013 + 32012 + 3(01)3

Here, the rst subscript indicates the appropriate random variable, the second subscript indicates the order of the moment, and the superscript is a prime (0) for raw moments (moments about the origin) and is unprimed for central moments (moments about the mean). The moments can be used on their own to provide approximations for probabilities of aggregate claims by matching the rst few model and sample moments.

EXAMPLE 9.3

The observed mean (and standard deviation) of the number of claims and the individual losses over the past 10 months are 6.7 (2.3) and 179,247 (52,141), respectively. Determine the mean and variance of aggregate claims per month.



E() = Var() =

Hence, the mean and standard deviation of aggregate claims are 1,200,955 and 433,797, respectively. ¤

EXAMPLE 9.4

(Example 9.3 continued) Using normal and lognormal distributions as ap- proximating distributions for aggregate claims, calculate the probability that claims will exceed 140% of expected costs. That is,

Pr( 140 × 1,200,955) = Pr( 1,681,337)

67(179,247) = 1,200,955

67(52,141)2 + (23)2(179,247)2 = 188180 × 1011

    

   

For the normal distribution

THE COMPOUND MODEL FOR AGGREGATE CLAIMS 217

Ã E() 1,681,337 1,200,955!

Pr(1681337) = Pr pVar() = Pr( 1107) = 1 (1107) = 0134

  

For the lognormal distribution, from Appendix A, the mean and second raw moment of the lognormal distribution are

E() = exp( + 1 2) and E(2) = exp(2 + 22) 2

Equating these to 1200955 × 106 and 188180 × 1011 + (1200955 × 106)2 = 163047 × 1012 and taking logarithms results in the following two equations in two unknowns:

+ 1 2 = 1399863 2 + 22 = 2811989. 2

From this = 1393731 and 2 = 01226361 Then

Pr( 1,681,337) = 1 ln 1,681,337 1393731 ̧ (01226361)05

= 1 (1135913) = 0128

The normal distribution provides a good approximation when E() is large. In particular, if has the Poisson, binomial, or negative binomial distribu- tion, a version of the central limit theorem indicates that, as , , or , respectively, goes to innity, the distribution of becomes normal. In this example, E() is small so the distribution of is likely to be skewed. In this case the lognormal distribution may provide a good approximation, although there is no theory to support this choice. ¤

EXAMPLE 9.5

(Group dental insurance) Under a group dental insurance plan covering em- ployees and their families, the premium for each married employee is the same regardless of the number of family members. The insurance company has compiled statistics showing that the annual cost (adjusted to current dol- lars) of dental care per person for the benets provided by the plan has the distribution in Table 9.1 (given in units of 25 dollars).

Furthermore, the distribution of the number of persons per insurance cer- ticate (i.e., per employee) receiving dental care in any year has the distrib- ution given in Table 9.2.

The insurer is now in a position to calculate the distribution of the cost per year per married employee in the group. The cost per married employee

is

Determine the pf of up to 525. Determine the mean and standard devi- ation of total payments per employee.

X8 () = ()

=0

433,797

       

   

218

AGGREGATE LOSS MODELS

Table 9.1 Loss distribution for Example 9.5.

()

 

1 2 3 4 5 6 7 8 9 10

Table 9.2

0.150 0.200 0.250 0.125 0.075 0.050 0.050 0.050 0.025 0.025

Frequency distribution for Example 9.5.

 

* 0  0.05
* 1  0.10
* 2  0.15
* 3  0.20
* 4  0.25
* 5  0.15
* 6  0.06
* 7  0.03
* 8  0.01

The distribution up to amounts of 525 is given in Table 9.3. For example, 3 (4) = 1 (1) 2 (3) + 1 (2) 2 (2). In general, pick two columns whose

 

superscripts sum to the superscript of the desired column (in this case, 1+2 = 3). Then add all combinations from these columns where the arguments sum to the desired argument (in this case 1+3 = 4 and 2+2 = 4). To obtain (), each row of the matrix of convolutions of () is multiplied by the probabilities from the row below the table and the products are summed. For example, (2) = 005(0) + 010(02) + 015(0225) = 02338.

The reader may wish to verify using (9.9) that the rst two moments of the distribution () are

E() = 1258 Var() = 587464

Hence the annual cost of the dental plan has mean 1258 × 25 = 31450 dollars and standard deviation 191.6155 dollars. (Why can’t the calculations be done from Table 9.3?) ¤

It is common for insurance to be oered in which a deductible is applied to the aggregate losses for the period. When the losses occur to a policyholder it is called insurance coverage and when the losses occur to an insurance company it is called

   

   

THE COMPOUND MODEL FOR AGGREGATE CLAIMS 219 Table 9.3 Aggregate probabilities for Example 9.5.

0 1 2 3 4 5 6 7 8 ()

 

* 0  1 0
* 1  0.150
* 2  0 .200
* 3  0 .250
* 4  0 .125
* 5  0 .075
* 6  0 .050
* 7  0 .050
* 8  0 .050
* 9  0 .025
* 10  0 .025
* 11  0 0
* 12  0 0
* 13  0 0
* 14  0 0
* 15  0 0
* 16  0 0
* 17  0 0
* 18  0 0
* 19  0 0
* 20  0 0
* 21  0 0

.05 .10

0 0

0 0 .02250 0 .06000 .00338 .11500 .01350 .13750 .03488 .13500 .06144 .10750 .08569 .08813 .09750 .07875 .09841 .07063 .09338 .06250 .08813 .04500 .08370 .03125 .07673 .01750 .06689 .01125 .05377 .00750 .04125 .00500 .03052 .00313 .02267 .00125 .01673 .00063 .01186 0 .00800

.15 .20

0 0 0 0 0 0 0 0

.00051 0 .00270 .00008 .00878 .00051 .01999 .00198 .03580 .00549 .05266 .01194 .06682 .02138 .07597 .03282 .08068 .04450 .08266 .05486 .08278 .06314 .08081 .06934 .07584 .07361 .06811 .07578 .05854 .07552 .04878 .07263 .03977 .06747 .03187 .06079

.25 .15

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

reinsurance coverage. The latter version is a common method for an insurance company to protect itself against an adverse year (as opposed to protecting against a single, very large claim). More formally, we present the following denition.

Denition 9.3 Insurance on the aggregate losses, subject to a deductible, is called stop-loss insurance. The expected cost of this insurance is called the net stop- loss premium and can be computed as E[( )+], where is the deductible and the notation (·)+ means to use the value in parentheses if it is positive but to use zero otherwise.

For any aggregate distribution, E[()+]=Z [1()]

If the distribution is continuous, the net stop-loss premium can be computed directly

from the denition as E[()+]=Z ()()

Similarly, for discrete random variables,

E[( )+] = X( )()

.00001 0 0 .00009 .00000 0 .00042 .00002 .00000 .00136 .00008 .00000 .00345 .00031 .00002 .00726 .00091 .00007 .01305 .00218 .00022 .02062 .00448 .00060 .02930 .00808 .00138 .03826 .01304 .00279 .04677 .01919 .00505 .05438 .02616 .00829 .06080 .03352 .01254 .06573 .04083 .01768 .06882 .04775 .02351 .06982 .05389 .02977

0 .05000 0.01500 .02338 .03468 .03258 .03579 .03981 .04356 .04752 .04903 .05190 .05138 .05119 .05030 .04818 .04576 .04281 .03938 .03575 .03197 .02832 .02479



.06 .03 .01

    

   

220 AGGREGATE LOSS MODELS Any time there is an interval with no aggregate probability, the following result

may simplify calculations.

Theorem9.4 SupposePr()=0. Then,for, E[( )+] = E[( )+] + E[( )+].

 

 That is, the net stop-loss premium can be calculated via linear interpolation.

Proof: From the assumption, Z() = (), . Then, E[( )+] = [1()]

ZZZ

= [1()] [1()]

= E[()+] [1()]

= E[( )+] ( )[1 ()] (9.10) Then, by setting = in (9.10),

and, therefore,

E[( )+] = E[( )+] ( )[1 ()]

1()= E[()+]E[()+]



Substituting this formula in (9.10) produces the desired result. ¤ Further simplication is available in the discrete case, provided places proba-

bility at equally spaced values.

Theorem9.5 AssumePr(=)= 0forsomexed0and=01 and Pr( = ) = 0 for all other . Then, provided = , with a nonnegative

integer

Proof:

E[()+]=X {1[(+)]} =0

E[()+] =

X()()

* =  X ( ) =  1
* =  X X
* =   {1 [( + )]} =0

= =0

= X X =0 =++1

¤

   

   

THE COMPOUND MODEL FOR AGGREGATE CLAIMS 221 In the discrete case with probability at equally spaced values, a simple recursion

holds. Corollary 9.6 Under the conditions of Theorem 9.5,

E{[ ( + 1)]+} = E[( )+] [1 ()] This result is easy to use because, when = 0, E[(0)+] = E() = E()E(),

which can be obtained directly from the frequency and severity distributions.

EXAMPLE 9.6

(Example 9.5 continued) The insurer is examining the eect of imposing an aggregate deductible per employee. Determine the reduction in the net pre- mium as a result of imposing deductibles of 25, 30, 50, and 100 dollars.

From Table 9.3, the cdf at 0, 25, 50, and 75 dollars has values 0.05, 0.065, 0.08838, and 0.12306. With E() = 25(1258) = 3145 we have

E[( 25)+] E[( 50)+] E[( 75)+]



= 3145 25(1 005) = 29075 = 29075 25(1 0065) = 267375 = 267375 25(1 008838) = 2445845 = 2445845 25(1 012306) = 222661

E[( 100)+] From Theorem 9.4, E[( 30)+] = 20 29075 + 5 267375 = 28607. When

 

25 25

compared to the original premium of 314.5, the reductions are 23.75, 28.43, 47.125, and 91.839 for the four deductibles. ¤

9.3.1 Exercises

9.5 From (9.8), show that the relationships between the moments in (9.9) hold. 9.6 (\*) When an individual is admitted to the hospital, the hospital charges have

the following characteristics:

Standard Charges Mean deviation

Room 1,000 500 Other 500 300

2. The covariance between an individual’s room charges and other charges is 100,000.

An insurer issues a policy that reimburses 100% for room charges and 80% for other charges. The number of hospital admissions has a Poisson distribution with parameter 4. Determine the mean and standard deviation of the insurer’s payout for the policy.

9.7 Aggregate claims have been modeled by a compound negative binomial dis- tribution with parameters = 15 and = 5. The claim amounts are uniformly



1.

     

   

222 AGGREGATE LOSS MODELS distributed on the interval (0 10). Using the normal approximation, determine the

premium such that the probability that claims will exceed premium is 0.05.

9.8 Automobile drivers can be divided into three homogeneous classes. The num- ber of claims for each driver follows a Poisson distribution with parameter . De- termine the variance of the number of claims for a randomly selected driver, using the following data.



Class

Table 9.4 Data for Exercise 9.8.

Proportion of population



1 0.25 5 2 0.25 3 3 0.50 2

9.9 (\*) Assume 1, 2, and 3 are mutually independent loss random variables with probability functions as given in Table 9.5. Determine the pf of = 1 + 2 +3.



Table 9.5 Distributions for Exercise 9.9.



0 1 2 3

1 ()

0.90 0.10 0.00 0.00

2 ()

0.50 0.30 0.20 0.00

3 ()

0.25 0.25 0.25 0.25

 

9.10 (\*) Assume 1, 2, and 3 are mutually independent random variables with probability functions as given in Table 9.6. If = 1 + 2 + 3 and (5) = 006, determine .



0 1 2 3

Table 9.6 Distributions for Exercise 9.10.

1 () 2 ()

0.6 1 0.2 0 0.1 0 0.1

3 ()

0.25 0.25 0.25 0.25

 

9.11 (\*) Consider the following information about AIDS patients: 1. The conditional distribution of an individual’s medical care costs, given that

the individual does not have AIDS, has mean 1,000 and variance 250,000.

2. The conditional distribution of an individual’s medical care costs, given that the individual does have AIDS, has mean 70,000 and variance 1,600,000.

   

   

THE COMPOUND MODEL FOR AGGREGATE CLAIMS 223 3. The number of individuals with AIDS in a group of randomly selected

adults has a binomial distribution with parameters and = 001.

An insurance company determines premiums for a group as the mean plus 10% of the standard deviation of the group’s aggregate claims distribution. The premium for a group of 10 independent lives for which all individuals have been proven not to have AIDS is . The premium for a group of 10 randomly selected adults is . Determine .

9.12 (\*) You have been asked by a city planner to analyze oce cigarette smok- ing patterns. The planner has provided the information in Table 9.7 about the distribution of the number of cigarettes smoked during a workday.

Table 9.7 Data for Exercise 9.12.

Male Female

Mean 6 3 Variance 64 31

The number of male employees in a randomly selected oce of employees has a binomial distribution with parameters and 0.4. Determine the mean plus the standard deviation of the number of cigarettes smoked during a workday in a randomly selected oce of eight employees.

9.13 (\*) For a certain group, aggregate claims are uniformly distributed over (010) Insurer A proposes stop-loss coverage with a deductible of 6 for a pre- mium equal to the expected stop-loss claims. Insurer B proposes group coverage with a premium of 7 and a dividend (a premium refund) equal to the excess, if any, of 7 over claims. Calculate such that the expected cost to the group is equal under both proposals.

9.14 (\*) For a group health contract, aggregate claims are assumed to have an exponential distribution where the mean of the distribution is estimated by the group underwriter. Aggregate stop-loss insurance for total claims in excess of 125% of the expected claims is provided by a gross premium that is twice the expected stop-loss claims. You have discovered an error in the underwriter’s method of calcu- lating expected claims. The underwriter’s estimate is 90% of the correct estimate. Determine the actual percentage loading in the premium.

9.15 (\*) A random loss, has the probability function given in Table 9.8. You are given that E() = 4 and E[( )+] = 2 Determine

9.16 (\*) A reinsurer pays aggregate claim amounts in excess of , and in return it receives a stop-loss premium E[( )+] You are given E[( 100)+] = 15 E[( 120)+] = 10, and the probability that the aggregate claim amounts are greater than 80 and less than or equal to 120 is 0. Determine the probability that the aggregate claim amounts are less than or equal to 80.

      

   

224 AGGREGATE LOSS MODELS

Table 9.8

0 1 2 3 4 5 6 7 8 9

Data for Exercise 9.15.

()

0.05 0.06 0.25 0.22 0.10 0.05 0.05 0.05 0.05 0.12

1 , 0 100. Two policies 100

  

9.17 (\*) A loss random variable has pdf () = can be purchased to alleviate the nancial impact of the loss.



( 0 50 = 50 50



 = 0100

and where and are the amounts paid when the loss is . Both policies have the

same net premium, that is, E() = E(). Determine .

9.18 (\*) For a nursing home insurance policy, you are given that the average length of stay is 440 days and 30% of the stays are terminated in the rst 30 days. These terminations are distributed uniformly during that period. The policy pays 20 per day for the rst 30 days and 100 per day thereafter. Determine the expected benets payable for a single stay.

9.19 (\*) An insurance portfolio produces claims, where



0 1 3

Pr( = )

0.5 0.4 0.1

 

Individual claim amounts have the following distribution:

()

1 0.9 10 0.1

Individual claim amounts and are mutually independent. Calculate the prob- ability that the ratio of aggregate claims to expected claims will exceed 30.

      

   

THE COMPOUND MODEL FOR AGGREGATE CLAIMS 225

9.20 (\*) A company sells group travel-accident life insurance with payable in the event of a covered individual’s death in a travel accident. The gross premium for a group is set equal to the expected value plus the standard deviation of the group’s aggregate claims. The standard premium is based on the following two assumptions:

1. All individual claims within the group are mutually independent.
2. 2(1 ) = 2,500, where is the probability of death by travel accident for an individual.

In a certain group of 100 lives, the independence assumption fails because three specic individuals always travel together. If one dies in an accident, all three are assumed to die. Determine the dierence between this group’s premium and the standard premium.

9.21 (\*) A life insurance company covers 16,000 lives for one-year term life insur- ance, as follows:



Benet Number amount covered

1 8,000 2 3,500 4 4,500

Probability of claim

0.025 0.025 0.025

 

All claims are mutually independent. The insurance company’s retention limit is 2 units per life. Reinsurance is purchased for 0.03 per unit. The probability that the insurance company’s retained claims, , plus cost of reinsurance will exceed 1,000 is

" E() # Pr pVar()

Determine using a normal approximation. 9.22 (\*) The probability density function of individual losses is

( 00231 ́ 0100 () = 100

0 elsewhere. The amount paid, , is 80% of that portion of the loss that exceeds a deductible

of 10. Determine E().

9.23 (\*) An individual loss distribution is normal with = 100 and 2 = 9. The distribution for the number of claims, , is given in Table 9.9. Determine the probability that aggregate claims exceed 100.

9.24 (\*) An employer self-insures a life insurance program with the following two characteristics:

1. Given that a claim has occurred, the claim amount is 2,000 with probability 0.4 and 3,000 with probability 0.6,

      

   

226 AGGREGATE LOSS MODELS

Table 9.9

0 1 2 3

Table 9.10

0 1 2 3 4

Distribution for Exercise 9.23.

Pr( = )

0.5 0.2 0.2 0.1

Distribution for Exercise 9.24.

()

116 14 38 14 116

     

2. The number of claims has the distribution given in Table 9.10.

The employer purchases aggregate stop-loss coverage that limits the employer’s annual claims cost to 5,000. The aggregate stop-loss coverage costs 1,472. Deter- mine the employer’s expected annual cost of the program, including the cost of stop-loss coverage.

9.25 (\*) The probability that an individual admitted to the hospital will stay days or less is 1 08 for = 012 . A hospital indemnity policy provides a xed amount per day for the 4th day through the 10th day (i.e., for a maximum of 7 days). Determine the percentage increase in the expected cost per admission if the maximum number of days paid is increased from 7 to 14.

9.26 (\*) The probability density function of aggregate claims, , is given by () = 34 1. The relative loading andpthe value are selected so that h i

Pr[(1+)E()]=Pr E()+ Var() =090 Calculate and .

9.27 (\*) An insurance policy reimburses aggregate incurred expenses at the rate of 80% of the rst 1,000 in excess of 100, 90% of the next 1,000, and 100% thereafter. Express the expected cost of this coverage in terms of = E[()+] for dierent values of .

9.28 (\*) The number of accidents incurred by an insured driver in a single year has a Poisson distribution with parameter = 2. If an accident occurs, the probability that the damage amount exceeds the deductible is 0.25. The number of claims and the damage amounts are independent. What is the probability that there will be no damages exceeding the deductible in a single year?

9.29 (\*) The aggregate loss distribution is modeled by an insurance company using an exponential distribution. However, the mean is uncertain. The company uses

    

   

THE COMPOUND MODEL FOR AGGREGATE CLAIMS 227 a uniform distribution (2,000,000 4,000,000) to express its view of what the mean

should be. Determine the expected aggregate losses.

9.30 (\*) A group hospital indemnity policy provides benets at a continuous rate of 100 per day of hospital connement for a maximum of 30 days. Benets for partial days of connement are prorated. The length of hospital connement in days, , has the following continuance (survival) function for 0 30:

1004 010

Pr( )= 0950035 1020

065002 2030

For a policy period, each member’s probability of a single hospital admission is 0.1 and of more than one admission is 0. Determine the pure premium per member, ignoring the time value of money.

9.31 (\*) Medical and dental claims are assumed to be independent with compound Poisson distributions as follows:



Claim type

Medical claims Dental claims

Claim amount distribution

Uniform (0 1,000) 2 Uniform (0 200) 3

 

Let be the amount of a given claim under a policy that covers both medical and dental claims. Determine E[( 100)+], the expected cost (in excess of 100) of any given claim.

9.32 (\*) For a certain insured, the distribution of aggregate claims is binomial with parameters = 12 and = 025. The insurer will pay a dividend, , equal to the excess of 80% of the premium over claims, if positive. The premium is 5. Determine E[].

9.33 (\*) The number of claims in one year has a geometric distribution with mean 1.5. Each claim has a loss of 100. An insurance pays 0 for the rst three claims in one year and then pays 100 for each subsequent claim. Determine the expected insurance payment per year.

9.34 (\*) A compound Poisson distribution has = 5 and claim amount distribution (100) = 080, (500) = 016, and (1,000) = 004. Determine the probability that aggregate claims will be exactly 600.

9.35 (\*) Aggregate payments have a compound distribution. The frequency dis- tribution is negative binomial with = 16 and = 6, and the severity distribution is uniform on the interval (0 8). Use the normal approximation to determine the premium such that the probability is 5% that aggregate payments will exceed the premium.

9.36 (\*) The number of losses is Poisson with = 3. Loss amounts have a Burr distribution with = 3, = 2, and = 1. Determine the variance of aggregate losses.

   

   

228 AGGREGATE LOSS MODELS 9.4 ANALYTIC RESULTS

For most choices of distributions of and the s, the compound distributional values can only be obtained numerically. Subsequent sections in this chapter are devoted to such numerical procedures.

However, for certain combinations of choices, simple analytic results are available, thus reducing the computational problems considerably.

EXAMPLE 9.7

(Compound negative binomial—exponential) Determine the distribution of when the frequency distribution is negative binomial with an integer value for the parameter and the severity distribution is exponential.

The mgf of is

() = [()]

* =   [(1 )1]
* =  {1 [(1 )1 1]} With a bit of algebra, this can be rewritten as  μ 1¶ ()= 1+1+{[1(1+)] 1}

 

which is of the form where

( ) = [ ( ) ]  ̧

()= 1+1+(1)



the pgf of the binomial distribution with parameters and (1 + ), and () is the mgf of the exponential distribution with mean (1 + )

This transformation reduces the computation of the distribution function to the nite sum, that is,

() = 1X μ¶μ ¶μ 1 ¶ =1 1+ 1+

X1 [1(1 + )1]1(1+)1 ×=0 ! .

When = 1, has a compound geometric distribution, and in this case the preceding formula reduces to

()=1 exp ̧ 0 1+ (1+)

Hence, Pr( = 0) = (0) = (1 + )1, and because () is dierentiable, i t h a s p d f ( ) = 0 ( ) , f o r 0 . T h a t i s , f o r 0 , h a s p d f

() = exp ̧ (1+)2 (1+)

          

   

To summarize, if = 1, has a point mass of (1+)1 at zero and an exponentially decaying density over the positive axis. This example arises again in Chapter 11 in connection with ruin theory. ¤

As is clear from Example 9.7, useful formulas may result with exponential claim sizes. The following example considers this case in more detail.

EXAMPLE 9.8

(Exponential severities) Determine the cdf of for any compound distribution with exponential severities.

The mgf of the sum of independent exponential random variables each with mean is

1+2+···+ () = (1 ) which is the mgf of the gamma distribution with cdf

() = 3; ́

(see Appendix A). For integer values of the values of (; ) can be calcu- lated exactly (see Appendix A for the derivation) as

ANALYTIC RESULTS 229

  

From (9.3)

1 X

(;)=1=0 ! =123 () = 0 + X 3; ́

(9.11)

(9.12)



=1  The density function can be obtained by dierentiation,

X 1 () = ()

(9.13) Returning to the distribution function, substituting (9.11) in (9.12) yields



() = 1 =1

(9.14)

(9.15)

¤

1

X

X () ! 0

( ) = 1 X ( ) X =0 ! =+1

X ̄ () = 1 ! , 0

=0 where ̄ = P=+1 for = 01 .

=1



=0 Interchanging the order of summation yields

     

   

230 AGGREGATE LOSS MODELS

For frequency distributions that assign positive probability to all nonnegative integers, (9.14) can be evaluated by taking sucient terms in the rst summation. For distributions for which Pr( ) = 0, the rst summation becomes nite. For example, for the binomial frequency distribution, (9.14) becomes

μ¶ 1 X X ()

()=1 (1) ! . (9.16) =1 =0



The approach of Example 9.8 may be extended to the larger class of mixed Erlang severity distributions, as shown in the next example.

EXAMPLE 9.9

Determine the distribution of for any compound distribution with mixed Erlang severities.

Consider a severity distribution that is a discrete mixture of gamma distri- butions with integer shape parameters (such gamma distributions are called Erlang distributions). The pdf for this distribution may be expressed as

() = X () =1

where () is the Erlang- pdf, 1

() = ( 1)! 0

The set of mixing weights { : = 1 2 } is a discrete probability distri- bution (they must be nonnegative and sum to one). The pdf () is very exible in terms of its possible shapes. In fact, as shown by Tijms [174, pp. 163—164], any nonnegative continuous distribution may be approximated with arbitrarily smallPerror by a distribution of this form.

Let () = =1 be the pgf of { : = 12}. The moment generating function of is

() = Z () = X Z () 0 =10

Because () is a gamma pdf, we know that its mgf is (1 ), 1

 

and, thus,

() = X (1 ) = [(1 )1] =1

From its denition, () is a pgf, and (1)1 is the mgf of an exponential random variable, and so () may be viewed as the mgf of a compound distribution with exponential severity distribution. Recall that we are working with a compound distribution with arbitrary frequency distribution and severity as previously dened. The mgf of is, therefore,

() = [()] = [(1)1]

   

   

where () = P=0 = [()] is the pgf of a discrete compound distri- bution with frequency distribution and a discrete severity distribution with pgf (). If is a member of the ( 0) or ( 1) classes of distributions, the probabilities 0 1 may be computed recursively.

The distribution of may thus be viewed as being of the compound form with exponential severities and a frequency distribution that is itself of com- pound form. The cdf of is then given by Example 9.8. That is,

X ̄ () () = 1 ! 0with

=0 ̄ = X

=+1

Comparing (9.13) and (9.15) shows that the density function is also of mixed Erlang form, that is,

X 1  () = () (9.17)

=1 Further properties of the mixed Erlang class may be found in Exercises

9.40 and 9.41. ¤

Another useful type of analytic result is now presented.

EXAMPLE 9.10

(Severity distributions closed under convolution) A distribution is said to be closed under convolution if adding i.i.d. members of a family produces another member of that family. Further assume that adding members of a family produces a member with all but one parameter unchanged and the remaining parameter is multiplied by . Determine the distribution of when the severity distribution has this property.

The condition means that, if (;) is the pf of each , then the pf of 1 +2 +···+ is (;). This means that

() = X (;)

=1 = X (;)

=1

eliminating the need to carry out evaluation of the convolution. Severity distributions that are closed under convolution include the gamma and inverse Gaussian distributions. See Exercise 9.37. ¤

The ideas of Example 9.10 may be extended from severity distributions to the compound Poisson model for the aggregate distribution. The following theorem

ANALYTIC RESULTS 231

      

   

232 AGGREGATE LOSS MODELS generalizes the results of Theorem 6.16 for a discrete severity distribution to distri-

butions with arbitrary support.

Theorem 9.7 Suppose that has a compound Poisson distribution with Poisson parameter and severity distribution with cdf () for = 1 2 . Suppose also that 12 are independent. Then = 1 + ··· + has a compound Poisson distribution with Poisson parameter = 1 + · · · + and severity distri- bution with cdf

()=X () =1

Proof: Let () be the mgf of () for = 12. Then, has mgf () = E( ) = exp{ [ () 1]}

and, by the independence of the s, has mgf



()

Y =1

Y =1

=

() = X

exp{[() 1]} X

= exp=1 ()1

= exp () =1



Because P () is the mgf of () = P (), () is a compound

 

=1  Poisson mgf and the result follows. ¤

The advantage of the use of Theorem 9.7 is of a computational nature. Consider a company with several lines of business, each with a compound Poisson model for aggregate losses. Or consider a group insurance contract in which each member has a compound Poisson model for aggregate losses. In each case we are interested in computing the distribution of total aggregate losses. The theorem implies that it is not necessary to compute the distribution for each line or group member and then determine the distribution of the sum. Instead, a weighted average of the loss severity distributions of each component may be used as a severity distribution and the Poisson parameters added to obtain the frequency distribution. Then a single aggregate loss distribution calculation is sucient. Note that this is purely a com- putational shortcut. The following example illustrates the theorem and Example 9.9.

EXAMPLE 9.11

A group of ten independent lives is insured for health benets. The th life has a compound Poisson distribution for annual aggregate losses with frequency parameter = and severity distribution with pdf

1 10 ()= 10(1)! 0

=1

     

   

ANALYTIC RESULTS 233 and therefore the severity distributions are of Erlang form. Determine the

distribution of total aggregate claims for the ten lives.

L e t = 1 + · · · + 1 0 = 5 5 a n d l e t = = 5 5 . Fr o m T h e o r e m 9 . 7 the total aggregate losses have a compound Poisson distribution with Poisson parameter and severity distribution with pdf given by

10 X 1 10

()= 10(1)! =1

This is exactly the setting of Example 9.9. The compound distribution has a Poisson “frequency distribution” with parameter 55 and a “severity distribution” that places probability 55 on the value .1 From Theorem 6.12, 0 = exp(55) and for = 12, the compound Poisson recursion simplies to



= () = 1

=1

1 m i n X( 1 0 )

! and X 1 1 0

2 10 X ̄ (10)



Finally,



() = 10(1)! =1

with ̄ =P=+1 =1P=0 ¤

9.4.1 Exercises

9.37 The following questions concern closure under convolution:

* (a)  Show that the gamma and inverse Gaussian distributions are closed un- der convolution. Show that the gamma distribution has the additional property mentioned in Example 9.10.
* (b)  Discrete distributions can also be used as severity distributions. Which of the distributions in Appendix B are closed under convolution? How can this information be used in simplifying calculation of compound probabilities of the form (6.34)?

9.38 A compound negative binomial distribution has parameters = 1, = 2, and severity distribution {(); = 012}. How do the parameters of the distribution change if the severity distribution is { () = ()[1 (0)]; = 1 2 } but the aggregate claims distribution remains unchanged?

1The words “frequency” and “severity” are in quotes because they are not referring to , the compound random variable of interest. The random variable is an articial compound variable used to solve this problem.

=0

    

   

234 AGGREGATE LOSS MODELS 9.39 Consider the compound logarithmic distribution with exponential severity

distribution.

* (a)  Show that the density of aggregate losses may be expressed as  1 X 1 ̧  () = ln(1 + ) =1 ! (1 + ) 1
* (b)  Reduce this to  () = exp{[(1 + )]} exp () ln(1+)

9.40 This exercise concerns generalized Erlang distributions.

* (a)  Prove that (1 1) = [(1 2)1], where  () = ̧ 1(1)  and = 21.
* (b)  Explain the implication of the identity in part (a) when is a positive  integerand02 1 .
* (c)  Consider the generalization of Example 3.7 (involving the distribution of the sum of independent exponentials with dierent means) where has a gamma distribution (not necessarily Erlang) with mgf () = (1 ) for = 12, and the s are independent. Then = 1 + · · · + has mgf  Y =1  and it is assumed, without loss of generality, that for = 12 1. If = 1 + ··· + is a positive integer, prove, using (a), that has a mixed Erlang distribution (Example 9.9) with mgf  where  with = for = 12 1, and describe the distribution with pgf (). Hence, show that has pdf

    

() =

(1 )

() = [(1 )1] 1 ̧

() = X = Y  = =1 1(1)



X () =

=

1  0.



(d) Describe how the results of Example 6.19 may be used to recursively compute the mixing weights +1 in part (c).

( 1)!

   

   

9.41 Consider the compound distribution with mixed Erlang severities in Example 9.9 with pdf () for 0 dened by

X () =

=1

() =

X =1

1 ( 1)!

0

ANALYTIC RESULTS 235



(a) Prove that

( + ) = X X ++1+1()+1() =0 =0

(b) Usepart(a)toprovethatfor0and0,

Z

X ( ) () ()= !

=0

and simplify it when = 1. (d) What happens to the result in (b) when = 0?

9.42 Consider the random sum = 1 +···+ (with = 0 if = 0). Let () be the df of and recall that E() = E()E() and () = [()].

* (a)  Let = Pr( ) = P=+1 , = 01 and dene 1 = E(), = 01 . Use Exercise 6.34 to show that {1 : = 0 1 } is a counting distribution with pgf  1 ( ) = X 1 = ( ) 1 =0 ( 1)E( )
* (b)  Let be independent of with equilibrium df ( ) = R 0 [ 1 ( ) ]  E() where () is the severity distribution. Let be the equilibrium ran-  dom variable associated with , with df ( ) = R 0 [ 1 ( ) ]  E( ) Use Exercise 3.28 to show that has mgf  () = ()1[ ()] and explain what this implies about the distribution of .



w h e r e

X =1

=  (c) Interpret the result in (b) in terms of stop-loss insurance when 0,

( + ) + ()

       

   

236

AGGREGATE LOSS MODELS

(c) If has the zero-modied geometric pf, = (1 0)(1 )1, = 12, prove, using (b) that

Z [1 ()] = 1 0 ZZ[1 ()] 1

 +1 [1 ( )][1 ()]

0

and interpret this result in terms of stop-loss insurance. (d) Use part (b) to prove that

Z [1 ()] = X 1⁄2Z [1 ()] =0

0

 

Z3⁄4 + [1()][1 ()]

9.5

The computation of the compound distribution function ( ) = X ( ) ( 9 . 1 8 )

or the corresponding probability (density) function is generally not an easy task, even in the simplest of cases. In this section we discuss a number of approaches to numerical evaluation of (9.18) for specic choices of the frequency and severity distributions as well as for arbitrary choices of one or both distributions.

One approach is to use an approximating distribution to avoid direct calculation of (9.18). This approach is used in Example 9.4 where the method of moments is used to estimate the parameters of the approximating distribution. The advantage of this method is that it is simple and easy to apply. However, the disadvantages are signicant. First, there is no way of knowing how good the approximation is. Choosing dierent approximating distributions can result in very dierent results, particularly in the right-hand tail of the distribution. Of course, the approximation should improve as more moments are used; but after four moments, one quickly runs out of distributions!

The approximating distribution may also fail to accommodate special features of the true distribution. For example, when the loss distribution is of the continuous type and there is a maximum possible claim (e.g., when there is a policy limit), the severity distribution may have a point mass (“atom” or “spike”) at the maximum. The true aggregate claims distribution is of the mixed type with spikes at integral multiples of the maximum corresponding to 123 claims at the maximum. These spikes, if large, can have a signicant eect on the probabilities near such multiples. These jumps in the aggregate claims distribution function cannot be replicated by a smooth approximating distribution.

The second method to evaluate (9.18) or the corresponding pdf is direct calcula- tion. The most dicult (or computer intensive) part is the evaluation of the -fold convolutions of the severity distribution for = 2 3 4 . In some situations,

COMPUTING THE AGGREGATE CLAIMS DISTRIBUTION

=0

   

   

COMPUTING THE AGGREGATE CLAIMS DISTRIBUTION 237

there is an analytic form–for example, when the severity distribution is closed under convolution, as dened in Example 9.10 and illustrated in Examples 9.7—9.9. Otherwise the convolutions need to be evaluated numerically using

()=Z (1)() () (9.19)

When the losses are limited to nonnegative values (as is usually the case), the range of integration becomes nite, reducing (9.19) to

Z (1) ()= ()() (9.20)

0

These integrals are written in Lebesgue—Stieltjes form because of possible jumps in the cdf () at zero and at other points.2 Evaluation of (9.20) usually requires numerical integration methods. Because of the rst term inside the integral, (9.20) needs to be evaluated for all possible values of . This approach quickly becomes technically overpowering.

As seen in Example 9.5, when the severity distribution is discrete, the calcula- tions reduce to numerous multiplications and additions. For continuous severities, a simple way to avoid these technical problems is to replace the severity distrib- ution by a discrete distribution dened at multiples 012 of some convenient monetary unit such as 1,000.

In practice, the monetary unit can be made suciently small to accommodate spikes at maximum insurance amounts. The spike must be a multiple of the mon- etary unit to have it located at exactly the right point. As the monetary unit of measurement becomes small, the discrete distribution function needs to approach the true distribution function. The simplest approach is to round all amounts to the nearest multiple of the monetary unit; for example, round all losses or claims to the nearest 1,000. More sophisticated methods are discussed later in this chapter.

When the severity distribution is dened on nonnegative integers 0 1 2 , cal- culating () for integral requires + 1 multiplications. Then, carrying out

these calculations for all possible values of and up to requires a number of

multiplications that are of order 3, written as (3), to obtain the distribution

of (9.18) for = 0 to = . When the maximum value, , for which the aggre-

gate claims distribution is calculated is large, the number of computations quickly

becomes prohibitive, even for fast computers. For example, in real applications

can easily be as large as 1,000 and requires about 109 multiplications. Further, if

Pr( = 0) 0 and the frequency distribution is unbounded, an innite number of

calculations is required to obtain any single probability. This is because () 0

for all and all , and so the sum in (9.18) contains an innite number of terms. When Pr( = 0) = 0, we have () = 0 for and so (9.18) will have no

more than + 1 positive terms. Table 9.3 provides an example of this latter case. Alternative methods to more quickly evaluate the aggregate claims distribution are discussed in Sections 9.6 and 9.8. The rst such method, the recursive method,

2Without going into the formal denition of the Lebesgue—Stieltjes integral, it suces to interpret ()() as to be evaluated by integrating ()() over those values for which has a continuous distribution and then adding () Pr( = ) over those points where Pr( = ) 0. This formulation allows for a single notation to be used for continuous, discrete, and mixed random

variables.

   

   

238 AGGREGATE LOSS MODELS

reduces the number of computations discussed previously to (2), which is a con- siderable savings in computer time, a reduction of about 99.9% when = 1,000 compared to direct calculation. However, the method is limited to certain fre- quency distributions. Fortunately, it includes all frequency distributions discussed in Chapter 6 and Appendix B.

The second such method, the inversion method, numerically inverts a transform, such as the characteristic function, using a general or specialized inversion software package. Two versions of this method are discussed in this chapter.

9.6 THE RECURSIVE METHOD

Suppose that the severity distribution () is dened on 0 1 2 representing multiples of some convenient monetary unit. The number represents the largest possible payment and could be innite. Further, suppose that the frequency distri- bution, , is a member of the (1) class and therefore satises

=μ+¶1 =234

Then the following result holds.

Theorem 9.8 For the ( 1) class,



[1 ( + )0]() + P( + )()( ) () = =1 ,

(9.21)

Proof: This result is identical to Theorem 6.13 with appropriate substitution of notation and recognition that the argument of () cannot exceed . ¤

Corollary 9.9 For the ( 0) class, the result (9.21) reduces to

P ( + ) () ( ) () = =1 (9.22)

Note that when the severity distribution has no probability at zero, the denomi- nator of (9.21) and (9.22) equals 1. Further, in the case of the Poisson distribution, (9.22) reduces to

X  ()= ()() =12 (9.23)

=1

The starting value of the recursive schemes (9.21) and (9.22) is (0) = [ (0)] following Theorem 6.14 with an appropriate change of notation. In the case of the Poisson distribution we have

(0) = [1(0)] Starting values for other frequency distributions are found in Appendix D.



1(0) noting that is notation for min( ).



1(0)

    

   

9.6.1 Applications to compound frequency models

When the frequency distribution can be represented as a compound distribution (e.g., Neyman Type A, Poisson—inverse Gaussian) involving only distributions from the ( 0) or ( 1) classes, the recursive formula (9.21) can be used two or more times to obtain the aggregate claims distribution. If the frequency distribution can be written as

() = 1[2()] then the aggregate claims distribution has pgf

which can be rewritten as

where

() = [()] = 1{2[ ()]}

() = 1[1 ()]

1 () = 2[ ()]

(9.24) (9.25)

Now (9.25) is the same form as an aggregate claims distribution. Thus, if 2() is

in the ( 0) or ( 1) class, the distribution of 1 can be calculated using (9.21).

The resulting distribution is the “severity” distribution in (9.25). Thus, a second application of (9.21) to (9.24) results in the distribution of .

The following example illustrates the use of this algorithm.

EXAMPLE 9.12

The number of claims has a Poisson—ETNB distribution with Poisson para- meter = 2 and ETNB parameters = 3 and = 02. The claim size distribution has probabilities 0.3, 0.5, and 0.2 at 0, 10, and 20, respectively. Determine the total claims distribution recursively.

In the preceding terminology, has pgf () = 1 [2()], where 1() and 2() are the Poisson and ETNB pgfs, respectively. Then the total dol- lars of claims has pgf () = 1 [1 ()], where 1 () = 2 [ ()] is a compound ETNB pgf. We will rst compute the distribution of 1. We have (in monetary units of 10) (0) = 03 (1) = 05, and (2) = 02. To use the compound ETNB recursion, we start with

THE RECURSIVE METHOD 239



1 (0)

= 2 [ (0)]

{1+[1(0)]} (1+) 1(1+) {1 + 3(1 03)}02 (1 + 3)02 1(1+3)02

=

The remaining values of 1 () may be obtained from (9.21) with replaced by 1. In this case we have



= = 016369



= 3 =075 =(021)=06 1+3

0 = 0 1 = 02(3) (1+3)02+1 (1+3)

= 046947

     

   

240 AGGREGATE LOSS MODELS Then (9.21) becomes

[046947 (07P5 06)(0)] () + =1 (075 06) ()1 ( )

1 () = = 060577 () + 129032 075 06



1 (X075)(03) 3 ́

1 = 031873 ©£ ¡1¢¤

()1 ( ) = 060577(05) + 129032 £075 06 ¡ 1 ¢¤ (05)(016369)



The rst few probabilities are

1 (1) 1 (2) 1 (3) 1 (4)

= 060577(02) + 129032 075 06 2 (05)(031873)

+ £075 06 ¡ 2 ¢¤ (02)(016369)a = 022002 ©£ 2 ¡1¢¤

= 129032 075 06 3 (05)(022002)

+ £075 06 ¡ 2 ¢¤ (02)(031873)a = 010686 ©£ 3 ¡1¢¤

= 129032 075 06 4 (05)(010686) + £075 06 ¡ 2 ¢¤ (02)(022002)a = 006692

4

=1

      

We now turn to evaluation of the distribution of with compound Poisson pgf

() = 1 [1 ()] = [1 ()1] Thus the distribution

{1(); = 012} becomes the “secondary” or “claim size” distribution in an application of the

compound Poisson recursive formula. Therefore,

(0) = (0) = [1 (0)1] = [1 (0)1] = 2(0163691) = 018775 The remaining probabilities may be found from the recursive formula

2 X ()= 1()() =12

=1 The rst few probabilities are



(1) = (2) = (3) =

(4) =

= 008696

2 ¡ 1 ¢ (031873)(018775) = 011968 ¡1¢ ¡¢



2 1 (031873)(011968) + 2 2 (022002)(018775) = 012076 ¡2¢ ¡2¢

 

2 1 (031873)(012076) + 2 2 (022002)(011968) 33

 

+2 ¡ 3 ¢ (010686)(018775) = 010090 3



2 ¡ 1 ¢ (031873)(010090) + 2 ¡ 2 ¢ (022002)(012076) 44

 

+2 ¡ 3 ¢ (010686)(011968) + 2 ¡ 4 ¢ (006692)(018775) 44

¤

     

   

When the severity distribution has a maximum possible value at , the compu- tations are speeded up even more because the sum in (9.21) will be restricted to at most nonzero terms. In this case, then, the computations can be considered to be of order ().

9.6.2 Underow/overow problems

The recursion (9.21) starts with the calculated value of ( = 0) = [(0)]. For large insurance portfolios, this probability is very small, sometimes smaller than the smallest number that can be represented on the computer and is then represented on the computer as zero and the recursion (9.21) fails. This problem can be overcome in several dierent ways (see Panjer and Willmot [137]). One of the easiest ways is to start with an arbitrary set of values for (0) (1) () such as (00001), where is suciently far to the left in the distribution so that the true value of () is still negligible. Setting to a point that lies six standard deviations to the left of the mean is usually sucient. Recursion (9.21) is used to generate values of the distribution with this set of starting values until the values are consistently less than (). The “probabilities” are then summed and divided by the sum so that the “true” probabilities add to 1. Trial and error will dictate how small should be for a particular problem.

Another method to obtain probabilities when the starting value is too small is to carry out the calculations for a subset of the portfolio. For example, for the Poisson distribution with mean , nd a value of = 2 so that the probability at zero is representable on the computer when is used as the Poisson mean. Equation (9.21) is now used to obtain the aggregate claims distribution when is used as the Poisson mean. If () is the pgf of the aggregate claims using Poisson mean , then () = [()]2. Hence one can obtain successively the distributions with pgfs [()]2, [()]4, [()]8 [()]2 by convoluting the result at each stage with itself. This approach requires an additional convolutions in carrying out the calculations but involves no approximations. It can be carried out for any frequency distributions that are closed under convolution. For the negative binomial distribution, the analogous procedure starts with = 2. For the binomial distribution, the parameter must be integer valued. A slight modication can be used. Let = b2c when b·c indicates the integer part of function. When the convolutions are carried out, one still needs to carry out the calculations using (9.21) for parameter 2. This result is then convoluted with the result of the convolutions. For compound frequency distributions, only the primary distribution needs to be closed under convolution.

9.6.3 Numerical stability

Any recursive formula requires accurate computation of values because each such value will be used in computing subsequent values. Recursive schemes suer the risk of errors propagating through all subsequent values and potentially blowing up. In the recursive formula (9.21), errors are introduced through rounding at each stage because computers represent numbers with a nite number of signicant digits. The question about stability is, “How fast do the errors in the calculations grow as the computed values are used in successive computations?” This work has

THE RECURSIVE METHOD 241

   

   

242 AGGREGATE LOSS MODELS

been done by Panjer and Wang [136]. The analysis is quite complicated and well beyond the scope of this book. However, some general conclusions can be made.

Errors are introduced in subsequent values through the summation

X μ+ ¶()() =1

in recursion (9.21). In the extreme right-hand tail of the distribution of , this sum is positive (or at least nonnegative), and subsequent values of the sum will be decreasing. The sum will stay positive, even with rounding errors, when each of the three factors in each term in the sum is positive. In this case, the recursive formula is stable, producing relative errors that do not grow fast. For the Poisson and negative binomial based distributions, the factors in each term are always positive.

However, for the binomial distribution, the sum can have negative terms because is negative, is positive, and is a positive function not exceeding 1. In this case, the negative terms can cause the successive values to blow up with alter- nating signs. When this occurs, the nonsensical results are immediately obvious. Although it does not happen frequently in practice, the reader should be aware of this possibility in models based on the binomial distribution.

9.6.4 Continuous severity

The recursive method as presented here requires a discrete severity distribution, while it is customary to use a continuous distribution for severity. In the case of continuous severities, the analog of the recursion (9.21) is an integral equation, the solution of which is the aggregate claims distribution.

Theorem 9.10 For the ( 1) class of frequency distributions and any continuous severity distribution with probability on the positive real line, the following integral equation holds:

()=1()+Z μ+¶()() (9.26) 0

The proof of this result is beyond the scope of this book. For a detailed proof, see Theorems 6.14.1 and 6.16.1 of Panjer and Willmot [138], along with the associated corollaries. They consider the more general () class of distributions, which allow for arbitrary modication of initial values of the distribution. Note that the initial term is 1(), not [1 (+)0]() as in (9.21). Also, (9.26) holds for members of the ( 0) class as well.

Integral equations of the form (9.26) are Volterra integral equations of the sec- ond kind. Numerical solution of this type of integral equation has been studied in the text by Baker [12]. This book considers an alternative approach for continuous severity distributions. It is to use a discrete approximation of the severity distrib- ution in order to use the recursive method (9.21) and avoid the more complicated methods of Baker [12].

9.6.5 Constructing arithmetic distributions

The easiest approach to construct a discrete severity distribution from a continuous one is to place the discrete probabilities on multiples of a convenient unit of mea-

     

   

surement , the span. Such a distribution is called arithmetic because it is dened on the nonnegative integers. In order to arithmetize a distribution, it is important to preserve the properties of the original distribution both locally through the range of the distribution and globally–that is, for the entire distribution. This should preserve the general shape of the distribution and at the same time preserve global quantities such as moments.

The methods suggested here apply to the discretization (arithmetization) of con- tinuous, mixed, and nonarithmetic discrete distributions.

9.6.5.1 Method of rounding (mass dispersal) Let denote the probability placed at , = 012 . Then set3

0 = Prμ¶=μ0¶ μ22¶

This method concentrates all the probability one-half span either side of and places it at . There is an exception for the probability assigned to 0 This, in eect, rounds all amounts to the nearest convenient monetary unit, , the span of the distribution. When the continuous severity distribution is unbounded, it is reasonable to halt the discretization process at some point once most all the probability has been accounted for. If the index for this last point is , then = 1 [( 05) 0]. With this method the discrete probabilities are never negative and sum to 1, ensuring that the resulting distribution is legitimate.

9.6.5.2 Method of local moment matching In this method we construct an arith- metic distribution that matches moments of the arithmetic and the true severity distributions. Consider an arbitrary interval of length , denoted by [ + ). We locate point masses 01··· at points , +··· + so that the rst moments are preserved. The system of + 1 equations reecting these conditions is

THE RECURSIVE METHOD 243

 

= Pr + 22

 

= μ+0¶μ0¶ =12 22

 

X Z + 0 ( +) = () =012 (9.27)

0

where the notation “0” at the limits of the integral indicates that discrete proba- bility at is to be included but discrete probability at + is to be excluded. Arrange the intervals so that +1 = + and so the endpoints coincide. Then the point masses at the endpoints are added together. With 0 = 0, the

=0

resulting discrete distribution has successive probabilities:

0 = 0 0 1 = 01 2 = 02 =0 +10 +1 =1 +2 =12

(9.28)

3 The notation ( 0) indicates that discrete probability at should not be included. For continuous distributions, this will make no dierence. Another way to look at this is that when there is discrete probability at one of the boundary points, it should be assigned to the value one-half span above that point.

   

   

244 AGGREGATE LOSS MODELS

By summing (9.27) for all possible values of , with 0 = 0, it is clear that the rst moments are preserved for the entire distribution and that the probabilities add to 1 exactly. It only remains to solve the system of equations (9.27).

Theorem 9.11 The solution of (9.27) is =Z +0Y () =01. (9.29)



0 6= ( ) Proof: The Lagrange formula for collocation of a polynomial () at points

01 is

X Y ()= ()



=0 6=  Applying this formula to the polynomial () = over the points , +

+ yields X Y

= ( +) =01. =0 6= ( )



Integrating over the interval [ +) with respect to the severity distribution

results in Z +0 0

X ( ) = ( + )

=0 where is given by (9.29). Hence, the solution (9.29) preserves the rst mo-

ments, as required. ¤ EXAMPLE 9.13

Suppose has the exponential distribution with pdf () = 0101. Use a span of = 2 to discretize this distribution by the method of rounding and by matching the rst moment.

For the method of rounding, the general formulas are

0 = (1) = 1 01(1) = 009516 = (2 + 1) (2 1) = 01(21) 01(2+1)

The rst few values are given in Table 9.11. For matching the rst moment we have = 1 and = 2. The key

equations become



0 1

and then

0

2+2 2

=

=

Z 2+2 2 2 01 01(2+2) 01(2) Z2 2 (01) = 5 4

2

1 + = 501(22) 1001(2) + 501(2+2) 10

 

= =

2 (01)01 = 601(2+2) + 501(2) 0 = 502 4 = 009365

   

   

THE RECURSIVE METHOD 245 Discretization of the exponential distribution by two methods.

Table 9.11

0 1 2 3 4 5 6 7 8 9 10 0.02711



rounding

matching

0.09365 0.16429 0.13451 0.11013 0.09017 0.07382 0.06044 0.04948 0.04051 0.03317 0.02716



0.09516 0.16402 0.13429 0.10995 0.09002 0.07370 0.06034 0.04940 0.04045 0.03311



The rst few values also are given in Table 9.11. A more direct solution for matching the rst moment is provided in Exercise 9.43. ¤

This method of local moment matching was introduced by Gerber and Jones [54] and Gerber [53] and further studied by Panjer and Lutek [135] for a variety of empirical and analytical severity distributions. In assessing the impact of errors on aggregate stop-loss net premiums (aggregate excess-of-loss pure premiums), Panjer and Lutek [135] found that two moments were usually sucient and that adding a third moment requirement adds only marginally to the accuracy. Furthermore, the rounding method and the rst-moment method ( = 1) had similar errors, while the second-moment method ( = 2) provided signicant improvement. The specic formulas for the method of rounding and the method of matching the rst moment are given in Appendix E. A reason to favor matching zero or one moment is that the resulting probabilities will always be nonnegative. When matching two or more moments, this cannot be guaranteed.

The methods described here are qualitatively similar to numerical methods used to solve Volterra integral equations such as (9.26) developed in numerical analysis (see, e.g., Baker [12]).

9.6.6 Exercises

9.43 Show that the method of local moment matching with = 1 (matching total probability and the mean) using (9.28) and (9.29) results in

0 = 1 E[ ]

=2E[]E[(1)]E[(+1)]

=12

 

and that {; = 0 1 2 } forms a valid distribution with the same mean as the original severity distribution. Using the formula given here, verify the formula given in Example 9.13.

   

   

246 AGGREGATE LOSS MODELS

9.44 You are the agent for a baseball player who desires an incentive contract that will pay the amounts given in Table 9.12. The number of times at bat has a Poisson distribution with = 200. The parameter is determined so that the probability of the player earning at least 4,000,000 is 0.95. Determine the player’s expected compensation.

Table 9.12 Data for Exercise 9.44.



Probability of hit Type of hit per time at bat

Single 0.14 Double 0.05 Triple 0.02 Home run 0.03

9.45 A weighted average of two Poisson distributions =11 +(1)22

Compensation per hit

2 3 4

   

! ! has been used by some authors, for example, Tröbliger [175], to treat drivers as

either “good” or “bad” (see Example 6.23).

* (a)  Find the pgf () of the number of losses in terms of the two pgfs 1()  and 2() of the number of losses of the two types of drivers.
* (b)  Let () denote a severity distribution dened on the nonnegative in- tegers. How can (9.23) be used to compute the distribution of aggregate claims for the entire group?
* (c)  Can your approach from (b) be extended to other frequency distribu- tions?

9.46 (\*) A compound Poisson aggregate loss model has ve expected claims per

year. The severity distribution is dened on positive multiples of 1,000. Given that

(1) = 5 and (2) = 55, determine (2). 2

9.47 (\*) For a compound Poisson distribution, = 6 and individual losses have pf (1) = (2) = (4) = 1 . Some of the pf values for the aggregate distribution

3

are given in Table 9.13. Determine (6).

 

Table 9.13

3 4 5 6 7

Data for Exercise 9.47.

()

0.0132 0.0215 0.0271

(6) 0.0410

      

   

9.48 Consider the ( 0) class of frequency distributions and any severity distrib- ution dened on the positive integers {1 2 }, where is the maximum possible single loss.

(a) Show that, for the compound distribution, the following backward re- cursion holds:

(+)P1 μ+ ¶( )(+) =1 +

THE RECURSIVE METHOD 247



()= μ¶ ++ ()

 

(b) For the binomial () frequency distribution, how can the preceding formula be used to obtain the distribution of aggregate losses? See Panjer and Wang [136].

9.49 (\*) Aggregate claims are compound Poisson with = 2 (1) = 1, and 4



(2) = 3 . For a premium of 6, an insurer covers aggregate claims and agrees to 4



pay a dividend (a refund of premium) equal to the excess, if any, of 75% of the premium over 100% of the claims. Determine the excess of premium over expected claims and dividends.

9.50 On a given day, a physician provides medical care to adults and children. Assume and have Poisson distributions with parameters 3 and 2, respectively. The distributions of length of care per patient are as follows:

Adult Child

1 hour 0.4 0.9 2 hour 0.6 0.1

Let , and the lengths of care for all individuals be independent. The physician charges 200 per hour of patient care. Determine the probability that the oce income on a given day is less than or equal to 800.

9.51 (\*) A group policyholder’s aggregate claims, , has a compound Poisson distribution with = 1 and all claim amounts equal to 2. The insurer pays the group the following dividend:

=1⁄26 6 0 6

Determine E[].

9.52 You are given two independent compound Poisson random variables, 1 and 2, where () = 12, are the two single-claim size distributions. You are given 1 = 2 = 1, 1(1) = 1, and 2(1) = 2(2) = 05. Let () be the single-claim size distribution function associated with the compound distribution = 1 + 2.

Calculate 4(6).

      

   

248 AGGREGATE LOSS MODELS 9.53 (\*) The variable has a compound Poisson claims distribution with the

following: 1. Individual claim amounts equal to 1, 2, or 3.

2. E() = 56. 3. Var() = 126. 4. =29. Determine the expected number of claims of size 2.

9.54 (\*) For a compound Poisson distribution with positive integer claim amounts, the probability function follows:

()= 1[016(1)+(2)+072(3)] =123

The expected value of aggregate claims is 168. Determine the expected number of claims.

9.55 (\*) For a portfolio of policies you are given the following:

1. The number of claims has a Poisson distribution.
2. Claim amounts can be 1, 2, or 3.
3. A stop-loss reinsurance contract has net premiums for various deductibles as given in Table 9.14.



Table 9.14

Deductible

4 5 6 7

Data for Exercise 9.55.

Net premium

0.20 0.10 0.04 0.02

  

Determine the probability that aggregate claims will be either 5 or 6.

9.56 (\*) For group disability income insurance, the expected number of disabilities per year is 1 per 100 lives covered. The continuance (survival) function for the length of a disability in days, , is

Pr()=1 =0110 10

The benet is 20 per day following a ve-day waiting period. Using a compound Poisson distribution, determine the variance of aggregate claims for a group of 1,500 independent lives.

    

   

THE IMPACT OF INDIVIDUAL POLICY MODIFICATIONS ON AGGREGATE PAYMENTS 249

9.57 A population has two classes of drivers. The number of accidents per indi- vidual driver has a geometric distribution. For a driver selected at random from Class I, the geometric distribution parameter has a uniform distribution over the interval (0 1). Twenty-ve percent of the drivers are in Class I. All drivers in Class II have expected number of claims 0.25. For a driver selected at random from this population, determine the probability of exactly two accidents.

9.58 (\*) A compound Poisson claim distribution has = 5 and individual claim amount distribution (5) = 06 and () = 04 where 5. The expected cost of an aggregate stop-loss insurance with a deductible of 5 is 28.03. Determine the value of .

9.59 (\*) Aggregate losses have a compound Poisson claim distribution with = 3 and individual claim amount distribution (1) = 04, (2) = 03, (3) = 02, and (4) = 01. Determine the probability that aggregate losses do not exceed 3.

9.60 Repeat Exercise 9.59 with a negative binomial frequency distribution with = 6 and = 05.

Note: Exercises 9.61 and 9.62 require the use of a computer.

9.61 A policy covers physical damage incurred by the trucks in a company’s eet. The number of losses in a year has a Poisson distribution with = 5. The amount of a single loss has a gamma distribution with = 05 and = 2,500. The insurance contract pays a maximum annual benet of 20,000. Determine the probability that the maximum benet will be paid. Use a span of 100 and the method of rounding.

9.62 An individual has purchased health insurance for which he pays 10 for each physician visit and 5 for each prescription. The probability that a payment will be 10 is 0.25, and the probability that it will be 5 is 0.75. The total number of payments per year has the Poisson—Poisson (Neyman Type A) distribution with 1 = 10 and 2 = 4. Determine the probability that total payments in one year will exceed 400. Compare your answer to a normal approximation.

9.63 Demonstrate that if the exponential distribution is discretized by the method of rounding, the resulting discrete distribution is a ZM geometric distribution.

9.7 THE IMPACT OF INDIVIDUAL POLICY MODIFICATIONS ON AGGREGATE PAYMENTS

In Section 8.6 the manner in which individual deductibles (both ordinary and fran- chise) aect both the individual loss amounts and the claim frequency distribution is discussed. In this section we consider the impact on aggregate losses. It is worth noting that both individual coinsurance and individual policy limits have an impact on the individual losses but not on the frequency of such losses, so we focus primar- ily on the deductible issues in what follows. We also remark that we continue to assume that the presence of policy modications does not have an underwriting im- pact on the individual loss distribution through an eect on the risk characteristics of the insured population, an issue discussed in Section 8.6. That is, the ground-up

   

   

250 AGGREGATE LOSS MODELS

distribution of the individual loss amount is assumed to be unaected by the policy modications, and only the payments themselves are aected.

From the standpoint of the aggregate losses, the relevant facts are now described. Regardless of whether the deductible is of the ordinary or franchise type, we shall as- sume that an individual loss results in a payment with probability . The individual ground-up loss random variable has policy modications (including deductibles) applied, so that a payment is then made. Individual payments may then be viewed on a per-loss basis, where the amount of such payment, denoted by , will be 0 if the loss results in no payment. Thus, on a per-loss basis, the payment amount is determined on each and every loss. Alternatively, individual payments may also be viewed on a per-payment basis. In this case, the amount of payment is denoted by , and on this basis payment amounts are only determined on losses that actually result in a nonzero payment being made. Therefore, by denition, Pr( = 0) = 0, and the distribution of is the conditional distribution of given that 0. Notationally, we write = | 0. Therefore, the cumulative distribution functions are related by

()=(1)+() 0

because 1 = Pr( = 0) = (0) (recall that has a discrete probability mass point 1 at 0, even if , and hence and have continuous probability density functions for 0). The moment generating functions of and are thus related by

()=(1)+ () (9.30) which may be restated in terms of expectations as

E()=E(| =0)Pr¡ =0¢+E(| 0)Pr¡ 0¢ It follows from Section 8.6 that the number of losses and the number of

payments are related through their probability generating functions by ()=(1+) (9.31)

where () = E3 ́ and () = E3 ́.

We now turn to the analysis of the aggregate payments. On a per-loss basis, the total payments may be expressed as = 1 +2 +···+ with = 0 if = 0 and where is the payment amount on the th loss. Alternatively, ignoring losses on which no payment is made, we may express the total payments on aper-paymentbasisas=1+2+···+ with=0if =0,and is the payment amount on the th loss, which results in a nonzero payment. Clearly, may be represented in two distinct ways on an aggregate basis. Of course, the moment generating function of on a per-loss basis is

()=E¡¢= [()] (9.32) whereas on a per-payment basis we have

() = E¡¢ = [ ()] (9.33)

Obviously, (9.32) and (9.33) are equal, as may be seen from (9.30) and (9.31). That is,

[()]= [1+ ()]= [ ()]

   

   

THE IMPACT OF INDIVIDUAL POLICY MODIFICATIONS ON AGGREGATE PAYMENTS 251

Consequently, any analysis of the aggregate payments may be done on either a per-loss basis (with compound representation (9.32) for the moment generating function) or on a per-payment basis (with (9.33) as the compound moment gen- erating function). The basis selected should be determined by whatever is more suitable for the particular situation at hand. While by no means a hard-and-fast rule, the authors have found it more convenient to use the per-loss basis to evaluate moments of . In particular, the formulas given in Section 8.5 for the individual mean and variance are on a per-loss basis, and the mean and variance of the aggre- gate payments may be computed using these and (9.9) but with replaced by andby.

If the (approximated) distribution of is of more interest than the moments, then a per-payment basis is normally to be preferred. The reason for this choice is that on a per-loss basis, underow problems may result if E() is large, and computer storage problems may occur due to the presence of a large number of zero probabilities in the distribution of , particularly if a franchise deductible is employed. Also, for convenience, we normally elect to apply policy modications to the individual loss distribution rst and then discretize (if necessary), rather than discretizing and then applying policy modications to the discretized distributions. This issue is only relevant if the deductible and policy limit are not integer multiples of the discretization span, however. The following example illustrates these ideas.

EXAMPLE 9.14

The number of ground-up losses is Poisson distributed with mean = 3. The individual loss distribution is Pareto with parameters = 4 and = 10. An individual ordinary deductible of 6, coinsurance of 75%, and an individual loss limit of 24 (before application of the deductible and coinsurance) are all applied. Determine the mean, variance, and distribution of aggregate payments.

We rst compute the mean and variance on a per-loss basis. The mean number of losses is E() = 3, and the mean individual payment on a per- loss basis is (using Theorem 8.7 with = 0 and the Pareto distribution)

E( ) = 075 [E( 24) E( 6)] = 075(32485 25195) = 054675 The mean of the aggregate payments is thus

() = E()E( ) = (3)(054675) = 164 The second moment of the individual payments on a per-loss basis is, using

Theorem 8.8 with = 0 and the Pareto distribution,

E £( )2¤ = (075)2{E £( 24)2¤ E £( 6)2¤ 2 2(6)E( 24) + 2(6)E( 6)}

= (075) [263790 105469 12(32485) + 12(25195)] = 398481

To compute the variance of aggregate payments, we do not need to explicitly determine Var( ) because is compound Poisson distributed, which implies

    

   

252 AGGREGATE LOSS MODELS

(using (6.41), e.g.) that Var() = E £( )2¤ = 3(398481) = 119544 = (346)2

To compute the (approximate) distribution of , we use the per-payment basis. First note that = ( 6) = [10(10 + 6)]4 = 015259, and the number of payments is Poisson distributed with mean E( ) = = 3(015259) = 045776. Let = 6| 6, so that is the individual payment random variable with only a deductible of 6. Then

Pr( ) = Pr( + 6) Pr( 6)



With coinsurance of 75%, = 075 has cumulative distribution function ()=1Pr(075 )=1 Pr( 6+075)



Pr( 6) That is, for less than the maximum payment of (075)(24 6) = 135,

()=Pr(6)(6+075) 135 ( 6)

and () = 1 for 135. We then discretize the distribution of (we thus apply the policy modications rst and then discretize) using a span of 2.25 and the method of rounding. This approach yields 0 = (1125) = 030124 1 = (3375) (1125) = 032768, and so on. In this situation care must be exercised in the evaluation of 6, and we have 6 = (14625) (12375)=1094126=005874. Then =11=0for=78. The approximate distribution of may then be computed using the compound Poisson recursive formula, namely, (0) = 045776(1030124) = 072625, and

045776 X6 ()= () =123

=1 Thus, (1) = (045776)(1)(032768)(072625) = 010894, for example. ¤

9.7.1 Exercises

9.64 Suppose that the number of ground-up losses has probability generat- ing function () = [( 1)], where is a parameter and is functionally independent of . The individual ground-up loss distribution is exponential with cumulative distribution function () = 1 0. Individual losses are subject to an ordinary deductible of and coinsurance of . Demonstrate that the aggregate payments, on a per-payment basis, have compound moment generating function given by (9.33), where has the same distribution as but with replaced by and has the same distribution as but with replaced by .

9.65 A ground-up model of individual losses has the gamma distribution with parameters = 2 and = 100. The number of losses has the negative binomial

     

   

INVERSION METHODS 253 distribution with = 2 and = 15. An ordinary deductible of 50 and a loss limit

of 175 (before imposition of the deductible) are applied to each individual loss.

* (a)  Determine the mean and variance of the aggregate payments on a per- loss basis.
* (b)  Determine the distribution of the number of payments.
* (c)  Determine the cumulative distribution function of the amount of a payment given that a payment is made.
* (d)  Discretizetheseveritydistributionfrom(c)usingthemethodofrounding and a span of 40.
* (e)  Use the recursive formula to calculate the discretized distribution of aggregate payments up to a discretized amount paid of 120.

9.8 INVERSION METHODS

An alternative to the recursive formula is the inversion method. This is another numerical approach and is based on the fact that there is a unique correspondence between a random variables distribution and its transform (such as the pgf, mgf, or cf). Compound distributions lend themselves naturally to this approach because their transforms are compound functions and are easily evaluated when both fre- quency and severity components are known. The pgf and cf of the aggregate loss distribution are

and

() = [()] () = E[] = [()] (9.34)

respectively. The characteristic function always exists and is unique. Conversely, for a given characteristic function, there always exists a unique distribution. The objective of inversion methods is to obtain the distribution numerically from the characteristic function (9.34).

It is worth mentioning that there has recently been much research in other areas of applied probability on obtaining the distribution numerically from the associated Laplace—Stieltjes transform. These techniques are applicable to the evaluation of compound distributions in the present context but are not discussed further here. A good survey is [2, pp. 257—323].

9.8.1 Fast Fourier transform

The fast Fourier transform (FFT) is an algorithm that can be used for inverting characteristic functions to obtain densities of discrete random variables. The FFT comes from the eld of signal processing. It was rst used for the inversion of characteristic functions of compound distributions by Bertram [18] and is explained in detail with applications to aggregate loss calculations by Robertson [149].

Denition 9.12 For any continuous function (), the Fourier transform is

the mapping

Z  ˜

() = ()

(9.35)

   

   

254 AGGREGATE LOSS MODELS The original function can be recovered from its Fourier transform as

1Z˜ () = 2 ()

˜ When () is a probability density function, () is its characteristic function.

˜ For our applications, () will be real valued. From (9.35), () is complex valued.

When () is a probability function of a discrete (or mixed) distribution, the de- nitions can be easily generalized (see, e.g., Fisz [47]). For the discrete case, the integrals become sums as given in the following denition.

Denition 9.13 Let denote a function dened for all integer values of that is

periodic with period length (i.e., + = for all ). For the vector (0 1 1),

the discrete Fourier transform is the mapping ˜ , = 101, dened



by

X1 μ2¶ ˜ = exp =101 (9.36)

=0



This mapping is bijective. In addition, ˜ is also periodic with period length . The

inverse mapping is

1 X 1 μ 2 ¶ = ˜exp =101

This inverse mapping recovers the values of the original function.

˜

transform as a bijective mapping of points into points. From (9.36), it is clear

that, to obtain values of ˜ , the number of terms that need to be evaluated is of

order 2, that is, (2). The fast Fourier transform (FFT) is an algorithm that reduces the number of

computations required to be of order ( ln2 ). This can be a dramatic reduction in computations when is large. The algorithm is not described here. The formulas and algorithms can be found in Press et al. [144] and is implemented in most computer packages including Excel°R . One requirement for using this method is that the vector of discrete severity probabilities must be the same length as the output vector and must be a power of 2.

In our applications, we use the FFT to invert the characteristic function when discretization of the severity distribution is done. The steps are:

1. Discretize the severity distribution using some methods such as those de- scribed in the previous section, obtaining the discretized severity distribution  (0)(1)(1) where = 2 for some integer and is the number of points desired in the  distribution () of aggregate claims.
2. Apply the FFT to this vector of values, obtaining (), the characteristic function of the discretized distribution. The result is also a vector of = 2 values.

=0

(9.37)

 

Because of the periodic nature of and , we can think of the discrete Fourier

   

   

1. Transform this vector using the pgf transformation of the claim frequency dis- tribution, obtaining () = [ ()], which is the characteristic function, that is, the discrete Fourier transform of the aggregate claims distribution, a vector of = 2 values.
2. Apply the Inverse Fast Fourier Transform (IFFT), which is identical to the FFT except for a sign change and a division by (see (9.37)). The result is a vector of length = 2 values representing the exact distribution of aggregate claims for the discretized severity model.

The FFT procedure requires a discretization of the severity distribution. When the number of points in the severity distribution is less than = 2, the severity distribution vector must be padded with zeros until it is of length .

When the severity distribution places probability on values beyond = , as is the case with most distributions discussed in Chapter 5, the probability that is missed in the right-hand tail beyond can introduce some minor error in the nal solution because the function and its transform are both assumed to be periodic with period , when, in reality, they are not. The authors suggest putting all the remaining probability at the nal point at = so that the probabilities add up to 1 exactly. Doing so allows for periodicity to be used for the severity distribution in the FFT algorithm and ensures that the nal set of aggregate probabilities will sum to 1. However, it is imperative that be selected to be large enough so that most all the aggregate probability occurs by the th point. The following example provides an extreme illustration.

EXAMPLE 9.15

Suppose the random variable takes on the values 1, 2, and 3 with proba- bilities 0.5, 0.4, and 0.1, respectively. Further suppose the number of claims has the Poisson distribution with parameter = 3. Use the FFT to obtain the distribution of using = 8 and = 4,096.

In either case, the probability distribution of is completed by adding one zero at the beginning (because places probability at zero, the initial representation of must also have the probability at zero given) and ei- ther 4 or 4,092 zeros at the end. The rst 8 results from employing the FFT and IFFT with = 4,096 appear in Table 9.15. The table shows the intermediate steps for the rst few calculations. For example, consider (5) = 099991176 00122713.4 Recall that for the Poisson distribution, () = exp[( 1)], and so

(099991176 00122713) = exp[3(099991176 00122713 1)] = exp(000026472)[cos(00368139) + sin(00368139)] = 0999058 0036796

using Euler’s formula. For the case = 8 is added in Table 9.16. The eight probabilities sum to 1 as they should. For the case = 4,096, the probabilities

4It is important to remember that we are not evaluating a function in the traditional sense. All 4,096 values of the FFT are found at once and the result depends on both the argument (5) and (4,096).

INVERSION METHODS 255

    

   

256 AGGREGATE LOSS MODELS Table 9.15

()

* 0  1
* 1  099999647 00024544
* 2  099998588 00049087
* 3  099996823 00073630
* 4  099994353 00098172
* 5  099991176 00122714
* 6  099987294 00147254
* 7  099982706 00171793

FFT calculations for = 4,096.

[ ()]

1 099996230 00073630 099984922 00147250 099966078 00220851 099939700 00294424 099905794 00367961 099864365 00441450 099815421 00514883

()

0.04979 0.07468 0.11575 0.13256 0.13597 0.12525 0.10558 0.08305

  

Table 9.16

0 1 2 3 4 5 6 7

Aggregate probabilities computed by the FFT and IFFT.



=8 ()

0.11227 0.11821 0.14470 0.15100 0.14727 0.13194 0.10941 0.08518

= 4,096 ()

0.04979 0.07468 0.11575 0.13256 0.13597 0.12525 0.10558 0.08305

 

also sum to 1, but there is not room here to show them all. It is easy to apply the recursive formula to this problem, which veries that all of the entries for = 4,096 are accurate to the ve decimal places presented. However, with = 8, the FFT gives values that are clearly distorted. If any generalization can be made, it is that more of the extra probability has been added to the smaller values of . ¤

9.8.2 Direct numerical inversion

The inversion of the characteristic function (9.34) has been done using approximate integration methods by Heckman and Meyers [64] in the case of Poisson, binomial, and negative binomial claim frequencies and continuous severity distributions. The method is easily extended to other frequency distributions.

In this method, the severity distribution function is replaced by a piecewise linear distribution. It further uses a maximum single-loss amount so the cdf jumps to 1 at the maximum possible individual loss. The range of the severity random variable is divided into intervals of possibly unequal length. The remaining steps parallel those of the FFT method. Consider the cdf of the severity distribution (),0. Let0=0 1 ··· bearbitrarilyselectedloss values. Then the probability that losses lie in the interval (1] is given by = ()(1). Using a uniform density over this interval results in the

   

   

approximating density function () = = ( 1) for 1 . Any remaining probability +1 = 1 ( ) is placed as a spike at . This approximating pdf is selected to make evaluation of the cf easy. It is not required for direct inversion. The cf of the approximating severity distribution is

() = Z () 0

X Z = ++1

=1 1

X 1 = ++1

=1 The cf can be separated into real and imaginary parts by using Euler’s formula

INVERSION METHODS 257



Then the real part of the cf is

() = Re[()] and the imaginary part is

() = Im[()]

1 X = [sin() sin(1)]

=1 ++1 cos()

1 X = [cos(1) cos()]

=1 ++1 sin()

=cos()+sin()

 

The cf of aggregate losses (9.34) is obtained as

() = [ ()] = [() + ()] which can be rewritten as

() = ()() The distribution of aggregate claims is obtained as

() = 1 + 1 Z () sinμ ¶ 20

because it is complex valued.

(9.38)

    

where is the standard deviation of the distribution of aggregate losses. Approx- imate integration techniques are used to evaluate (9.38) for any value of . The reader is referred to Heckman and Meyers [64] for details. They also obtain the net stop-loss (excess pure) premium for the aggregate loss distribution as

(9.39)

() = E[()+]=Z ()() Z () μ¶ μ ¶ ̧

=0 2 coscos

    

+ 2

    

   

258 AGGREGATE LOSS MODELS

from (9.38), where is the mean of the aggregate loss distribution and is the deductible.

Equation (9.38) provides only a single value of the distribution, while (9.39) provides only one value of the premium, but it does so quickly. The error of approximation depends on the spacing of the numerical integration method but is controllable.

9.8.3 Exercise

9.66 Repeat Exercises 9.61 and 9.62 using the inversion method. 9.9 CALCULATIONS WITH APPROXIMATE DISTRIBUTIONS

Whenever the severity distribution is calculated using an approximate method, the result is, of course, an approximation to the true aggregate distribution. In par- ticular, the true aggregate distribution is often continuous (except, perhaps, with discrete probability at zero or at an aggregate censoring limit) while the approxi- mate distribution may be any of the following:

* •  Discrete with probability at equally spaced values (as with recursion and FFT),
* •  discrete with probability 1 at arbitrary values (as with simulation, see Chapter 21), or
* •  a piecewise linear distribution function (as with Heckman—Meyers).  In this section we introduce reasonable ways to obtain values of () and E[( )] from those approximating distributions. In all cases we assume that the true distribution of aggregate payments is continuous, except, perhaps, with discrete probability at = 0.

9.9.1 Arithmetic distributions

For recursion and the FFT, the approximating distribution can be written as

0 1 , where = Pr( = ) and refers to the approximating distrib-

ution. While several methods of undiscretizing this distribution are possible, we

introduce only one. It assumes we can obtain 0 = Pr( = 0), the true probability

that aggregate payments are zero. The method is based on constructing a contin-

uous approximation to by assuming the probability is uniformly spread over

theinterval(1)to(+1)for=12. Fortheintervalfrom0to2,a 22

discrete probability of 0 is placed at zero and the remaining probability, 0 0 is spread uniformly over the interval. Let be the random variable with this mixed distribution. All quantities of interest are then computed using .

EXAMPLE 9.16

Let have the geometric distribution with = 2 and let have the exponen- tial distribution with = 100. Use recursion with a span of 2 to approximate the aggregate distribution and then obtain a continuous approximation.

      

   

0

1

2

3

4

5

6

7

8

9

10

()

0 0.009934 2 0.019605 4 0.019216 6 0.018836 8 0.018463 10 0.018097 12 0.017739 14 0.017388 16 0.017043 18 0.016706 20 0.016375

=()

0.335556 0.004415 0.004386 0.004356 0.004327 0.004299 0.004270 0.004242 0.004214 0.004186 0.004158

Table 9.17

CALCULATIONS WITH APPROXIMATE DISTRIBUTIONS 259 Discrete approximation to the aggregate payments distribution.

  

The exponential distribution was discretized using the method that pre-

serves the rst moment. The probabilities appear in Table 9.17. Also pre-

sented are the aggregate probabilities computed using the recursive formula.

We also note that 0 = Pr( = 0) = (1+)1 = 1. For = 12 the con- 3

tinuous approximation has pdf () = (2)2 2 1 2 + 1. We also have Pr( = 0) = 1 and () = (0335556 1 )1 = 0002223 0 1. 3 3 ¤

  

Returning to the original problem, it is possible to work out the general formulas for the basic quantities. For the cdf,

() = 0 +Z 0 0 0 2

= 0+2(00) 0 2

  

and

X1 Z () = +



=0 (12) X 1 ( 1 2 ) μ 1 ¶ μ 1 ¶

= + 2+2 =0

  

For the limited expected value (LEV),

Z E[( )] = 00 + 0 0 +[1()]

= 2+1(00)+[1()] 0 ( + 1) 2



0 2

     

   

260 AGGREGATE LOSS MODELS and

Z 2 E[()] = 0Z+ 0 0+

=

For = 1, the preceding formula reduces to

X 1 Z ( + 1 2 )

( 12)  (2)(0 0) + X [( + 12)+1 ( 12)+1]

0

0 2

 

=1 (12) + +[1()]



1

+1 =1 +1 +1 [( 12)]+1

 

+ ( + 1)  +[1()] μ1¶μ+1¶

  

2 (10) (00) 0

22

 

2 1 2 2

(00)+P+ [(12)]  4 =1 μ ¶2 μ ¶

E()=

These formulas are summarized in Appendix E.

EXAMPLE 9.17

(Example 9.16 continued) Compute the cdf and LEV at integral values from 1 to 10 using , , and the exact distribution of aggregate losses.

The exact distribution is available for this example. It was developed in Example 9.7 where it was determined that Pr( = 0) = (1 + )1 = 1 and

(9.40)

 

+[1()] 1 +1 22

   

the pdf for the continuous part is

3

() = exp ̧ = 2 300 0 (1+)2 (1+) 900

Thus we have

()=1+Z 2300=12300 30900 3

0 900 3 The requested values are given in Table 9.18.

     

and E()=Z 2 300+2300 =200(1300)

 

¤

   

   

CALCULATIONS WITH APPROXIMATE DISTRIBUTIONS 261 Table 9.18 Comparison of true aggregate payment values and two approximations.

  

cdf LEV

0.66556 1.32890 1.99003 2.64896 3.30570 3.96025 4.61263 5.26284 5.91088 6.55676



* 1  0.335552
* 2  0.337763
* 3  0.339967
* 4  0.342163
* 5  0.344352
* 6  0.346534
* 7  0.348709
* 8  0.350876
* 9  0.353036
* 10  0.355189

0.335556 0.339971 0.339971 0.344357 0.344357 0.348713 0.348713 0.353040 0.353040 0.357339

0.335556 0.337763 0.339970 0.342163 0.344356 0.346534 0.348712 0.350876 0.353039 0.355189

0.66556 0.66444 1.32890 1.32889 1.99003 1.98892 2.64897 2.64895 3.30571 3.30459 3.96027 3.96023 4.61264 4.61152 5.26285 5.26281 5.91089 5.90977 6.55678 6.55673



9.9.2 Empirical distributions

When the approximate distribution is obtained by simulation (the simulation process is discussed in Chapter 21), the result is an empirical distribution. Unlike approx- imations produced by recursion or the FFT, simulation does not place the prob- abilities at equally spaced values. As a result, it less clear how the approximate distribution should be smoothed. With simulation usually involving tens or hun- dreds of thousands of points, the individual points are likely to be close to each other. For these reasons it seems sucient to simply use the empirical distribution as the answer. That is, all calculations should be done using the approximate em- pirical random variable, . The formulas for the commonly required quantities are very simple. Let 1 2 be the simulated values. Then



and

() = number of

E[( )]= 1 X +[1()]

 

EXAMPLE 9.18

(Example 9.16 continued) Simulate 1,000 observations from the compound model with geometric frequency and exponential severity. Use the results to obtain values of the cdf and LEV for the integers from 1 to 10. The small sample size was selected so that only about 30 values between zero and 10 (not including zero) are expected.

The simulations produced an aggregate payment of zero 331 times. The set of nonzero values that were less than 10 plus the rst value past 10 are pre- sented in Table 9.19. Other than zero, none of the values appeared more than once in the simulation. The requested values from the empirical distribution along with the true values are given in Table 9.20. ¤

   

   

262 AGGREGATE LOSS MODELS Table 9.19 Simulated values of aggregate losses.



1—331 0 332 0.04 333 0.12 334 0.89 335 1.76 336 2.16 337 3.13 338 3.40 339 4.38 340 4.78 341 4.95 342 5.04 343 5.07 344 5.81 345 5.94

346 6.15 347 6.26 348 6.58 349 6.68 350 6.71 351 6.82 352 7.76 353 8.23 354 8.67 355 8.77 356 8.85 357 9.18 358 9.88 359 10.12

 

Table 9.20

()

* 0  0.331
* 1  0.334
* 2  0.335
* 3  0.336
* 4  0.338
* 5  0.341
* 6  0.345
* 7  0.351
* 8  0.352
* 9  0.356
* 10  0.358

9.9.3 Piecewise linear cdf

E( ) 0.0000

0.6656 1.3289 1.9900 2.6490 3.3057 3.9603 4.6126 5.2629 5.9109 6.5568

Empirical and smoothed values from a simulation.

() E() 0.333 0.0000

0.336 0.6671 0.338 1.3328 0.340 1.9970 0.342 2.6595 0.344 3.3206 0.347 3.9775 0.349 4.6297 0.351 5.2784 0.353 5.9250 0.355 6.5680

  

When using the Heckman—Meyers inversion method, the output is approximate values of the cdf () at any set of desired values. The values are approximate because the severity distribution function is required to be piecewise linear and because approximate integration is used. Let # denote an arbitrary random vari- able with cdf values as given by the Heckman—Meyers method at arbitrarily selected points0=1 2 ··· andlet =#(). Also,set =1sothat no probability is lost. The easiest way to complete the description of the smoothed distribution is to connect these points with straight lines. Let ## be the random variable with this particular cdf. Intermediate values of the cdf of ## are found by interpolation.

##()= (1) +( )1 1

    

   

CALCULATIONS WITH APPROXIMATE DISTRIBUTIONS 263 The formula for the limited expected value is (for 1 )

## E[( )]=

X Z 1 Z 1



=21 1

 + 1 +[1##()]



1 X1 (+1 +1)( 1)

1 = 1



=2 ( + 1)( 1)

+ +

( + 1)( 1) 1 (1) +( )1

̧

(+1 +1)( 1) 1

 

and when = 1, E(## )

9.9.4 Exercises

1 = X1 ( + 1)( 1) + (2 21)( 1)

=2 2 2( 1) +1 (1) +( )1 ̧.

1

  

9.67 Let the frequency (of losses) distribution be negative binomial with = 2 and = 2. Let the severity distribution (of losses) have the gamma distribution with = 4 and = 25. Determine (200) and E( 200) for an ordinary per-loss deductible of 25. Use the recursive formula to obtain the aggregate distribution and use a discretization interval of 5 with the method of rounding to discretize the severity distribution.

9.68 (Exercise 9.61 continued) Recall that the number of claims has a Poisson distribution with = 5 and the amount of a single claim has a gamma distribu- tion with = 05 and = 2,500. Determine the mean, standard deviation, and 90th percentile of payments by the insurance company under each of the following coverages. Any computational method may be used.

* (a)  A maximum aggregate payment of 20,000.
* (b)  A per-claim ordinary deductible of 100 and a per claim maximum pay- ment of 10,000. There is no aggregate maximum payment.
* (c)  A per-claim ordinary deductible of 100 with no maximum payment. There is an aggregate ordinary deductible of 15,000, an aggregate coin- surance factor of 0.8, and a maximum insurance payment of 20,000. This scenario corresponds to an aggregate reinsurance provision.

9.69 (Exercise 9.62 continued) Recall that the number of payments has the Poisson— Poisson distribution with 1 = 10 and 2 = 4, while the payment per claim by the insured is 5 with probability 0.75 and 10 with probability 0.25. Determine the

   

   

264 AGGREGATE LOSS MODELS expected payment by the insured under each of the following situations. Any com-

putational method may be used.

(a) A maximum payment of 400.

(b) A coinsurance arrangement where the insured pays 100% up to an ag- gregate total of 300 and then pays 20% of aggregate payments above 300.

9.10 COMPARISON OF METHODS

The recursive method has some signicant advantages. The time required to com- pute an entire distribution of points is reduced to (2) from (3) for the direct convolution method. Furthermore, it provides exact values when the severity distri- bution is itself discrete (arithmetic). The only source of error is in the discretization of the severity distribution. Except for binomial models, the calculations are guar- anteed to be numerically stable. This method is very easy to program in a few lines of computer code. However, it has a few disadvantages. The recursive method only works for the classes of frequency distributions described in Chapter 6. Using dis- tributions not based on the ( 0) and ( 1) classes requires modication of the formula or developing a new recursion. Numerous other recursions have recently been developed in the actuarial and statistical literature.

The FFT method is easy to use in that it uses standard routines available with many software packages. It is faster than the recursive method when is large because it requires calculations of order ln2 rather than 2. However, if the severity distribution has a xed (and not too large) number of points, the recursive method will require fewer computations because the sum in (9.21) will have at most terms, reducing the order of required computations to be of order , rather than 2 in the case of no upper limit of the severity. The FFT method can be extended to the case where the severity distribution can take on negative values. Like the recursive method, it produces the entire distribution.

The direct inversion method has been demonstrated to be very fast in calculat- ing a single value of the aggregate distribution or the net stop-loss (excess pure) premium for a single deductible . However, it requires a major computer pro- gramming eort. It has been developed by Heckman and Meyers [64] specically for ( 0) frequency models. It is possible to generalize the computer code to han- dle any distribution with a pgf that is a relatively simple function. This method is much faster than the recursive method when the expected number of claims is large. The speed does not depend on the size of in the case of the Poisson fre- quency model. In addition to being complicated to program, the method involves approximate integration whose errors depend on the method and interval size.

Through the use of transforms, both the FFT and inversion methods are able to handle convolutions eciently. For example, suppose a reinsurance agreement was to cover the aggregate losses of three groups, each with unique frequency and severity distributions. If = 1 2 3, are the aggregate losses for each group, the characteristic function for the total aggregate losses = 1 + 2 + 3 is () = 1 ()2 ()3 (), and so the only extra work is some multiplications prior to the inversion step. The recursive method does not accommodate convolutions as easily.

   

   

The Heckman—Meyers method has some technical diculties when being applied to severity distributions that are of the discrete type or have some anomalies, such as heaping of losses at some round number (e.g., 1,000,000). At any jump in the severity distribution function, a very short interval containing the jump needs to be dened in setting up the points (1 2 ).

We save a discussion of simulation for last because it diers greatly from the other methods. For those not familiar with this method, an introduction is provided in Chapter 21. The major advantage is a big one. If you can carefully articulate the model, you should be able to obtain the aggregate distribution by simulation. The programming eort may take a little time but can be done in a straightforward manner. Today’s computers will conduct the simulation in a reasonable amount of time. Most of the analytic methods were developed as a response to the excessive computing time that simulations used to require. That is less of a problem now. However, it is dicult to write a general-purpose simulation program. Instead, it is usually necessary to write a new routine as each problem occurs. Thus it is probably best to save the simulation approach for those problems that cannot be solved by the other methods. Then, of course, it is worth the eort because there is no alternative.

One other drawback of simulation occurs in extremely low frequency situations (which is where recursion excels). For example, consider an individual excess-of- loss reinsurance in which reinsurance benets are paid on individual losses above 1,000,000, an event that occurs about 1 time in 100, but when it does, the tail is extremely heavy (e.g., a Pareto distribution with small ). The simulation will have to discard 99% of the generated losses and then will need a large number of those that exceed the deductible (due to the large variation in losses). It may take a long time to obtain a reliable answer. One possible solution for simulation is to work with the conditional distribution of the loss variable, given that a payment has been made.

No method is clearly superior for all problems. Each method has both advantages and disadvantages when compared with the others. What we really have is an embarrassment of riches. Twenty-ve years ago, actuaries wondered if there would ever be eective methods for determining aggregate distributions. Today we can choose from several.

9.11 THE INDIVIDUAL RISK MODEL 9.11.1 The model

The individual risk model represents the aggregate loss as a xed sum of indepen- dent (but not necessarily identically distributed) random variables:

= 1 + 2 + · · · +

This formula is usually thought of as the sum of the losses from insurance contracts, for example, persons covered under a group insurance policy.

The individual risk model was originally developed for life insurance in which the probability of death within a year is and the xed benet paid for the death of the th person is . In this case, the distribution of the loss to the insurer for

THE INDIVIDUAL RISK MODEL 265

   

   

266 AGGREGATE LOSS MODELS thethpolicyis

1⁄2 1 =0 () = = .

The mean and variance of aggregate losses are

and

X =1

Y =1

2 ( 1 ) () = (1 + ) (9.41)

E() =

X =1

V a r ( ) = because the s are assumed to be independent. Then, the pgf of aggregate losses

is

In the special case where all the risks are identical with = and = 1, the pgf reduces to

() = [1 + ( 1)]

and in this case has a binomial distribution. The individual risk model can be generalized as follows. Let = , where

1 1 are independent. The random variable is an indicator vari- able that takes on the value 1 with probability and the value 0 with probability 1 . This variable indicates whether the th policy produced a payment. The random variable can have any distribution and represents the amount of the payment in respect of the th policy given that a payment was made. In the life insurance case, is degenerate, with all probability on the value .

The mgf corresponding to (9.41) is

Y () = [1 + ()]

=1 If we let = E() and 2 = Var(), then

(9.42)

(9.43)

(9.44)

and

Var()=

[2 +(1)2]

X =1

X

E() =

=1

You are asked to verify these formulas in Exercise 9.70. The following example is a simple version of this situation.

   

   

E() = 50(001)(65,000) + 25(001)(97,500) = 56,875

and

With regard to calculating the probabilities, there are at least three options. One is to do an exact calculation, which involves numerous convolutions and almost always requires more excessive computing time. Recursive formulas have been developed, but they are cumbersome and are not presented here. See De Pril [32] for one such method. One alternative is a parametric approximation as discussed for the collective risk model. Another alternative is to replace the individual risk model with a similar collective risk model and then do the calculations with that model. These two approaches are presented here.

9.11.2 Parametric approximation

A normal, gamma, lognormal, or any other distribution can be used to approximate the distribution, usually done by matching the rst few moments. Because the normal, gamma, and lognormal distributions each have two parameters, the mean and variance are sucient.

EXAMPLE 9.20

(Group life insurance) A small manufacturing business has a group life in- surance contract on its 14 permanent employees. The actuary for the insurer has selected a mortality table to represent the mortality of the group. Each employee is insured for the amount of his or her salary rounded up to the next 1,000 dollars. The group’s data are given in Table 9.21.

If the insurer adds a 45% relative loading to the net (pure) premium, what are the chances that it will lose money in a given year? Use the normal and lognormal approximations.

Var() = 50(001)(525,000,000) + 50(001)(099)(65,000)2 +25(001)(1,181,250,000) + 25(001)(099)(97,500)2

= 5,001,984,375. ¤

THE INDIVIDUAL RISK MODEL 267



EXAMPLE 9.19

Consider a group life insurance contract with an accidental death benet. Assume that for all members the probability of death in the next year is 0.01 and that 30% of deaths are accidental. For 50 employees, the benet for an ordinary death is 50,000 and for an accidental death it is 100,000. For the remaining 25 employees, the benets are 75,000 and 150,000, respectively. Develop an individual risk model and determine its mean and variance.

For all 75 employees = 001. For 50 employees, takes on the value 50,000 with probability 0.7 and 100,000 with probability 0.3. For them, = 65,000 and 2 = 525,000,000. For the remaining 25 employees takes on the value 75,000 with probability 0.7 and 150,000 with probability 0.3. For them, = 97,500 and 2 = 1,181,250,000. Then

    

   

268 AGGREGATE LOSS MODELS Table 9.21

Employee, Age (years) Sex

* 1  20 M 15,000
* 2  23 M 16,000
* 3  27 M 20,000
* 4  30 M 28,000
* 5  31 M 31,000
* 6  46 M 18,000
* 7  47 M 26,000
* 8  49 M 24,000
* 9  64 M 60,000
* 10  17 F 14,000
* 11  22 F 17,000
* 12  26 F 19,000
* 13  37 F 30,000
* 14  55 F 55,000

Employee data for Example 9.20.



Benet, Mortality rate,

0.00149 0.00142 0.00128 0.00122 0.00123 0.00353 0.00394 0.00484 0.02182 0.00050 0.00050 0.00054 0.00103 0.00479

 

Total

373,000

The mean and variance of the aggregate losses for the group are

X14

=1 X14



and

V a r ( ) =

2 ( 1 ) = 1 0 2 5 3 4 × 1 0 8

E()=

=2,05441

=1

The premium being charged is 145 × 2 05441 = 2,97889. For the normal approximation (in units of 1,000), the mean is 205441 and the variance is 102534. Then the probability of a loss is

Pr( 297889) = = Pr( 00913)

Pr 297889 205441 ̧ (102534)12



= 046 or 46% For the lognormal approximation (as in Example 9.4)

+ 1 2 = ln 205441 = 0719989 2



and From this = 0895289 and 2 = 3230555. Then

2 + 22 = ln(102534 + 2054412) = 4670533

Pr( 297889) =

1 ln 297889 + 0895289 ̧ (3230555)12



* =  1 (1105)
* =  013 or 13% ¤

   

   

9.11.3 Compound Poisson approximation

Because of the computational complexity of calculating the distribution of total claims for a portfolio of risks using the individual risk model, it has been popular to attempt to approximate the distribution by using the compound Poisson distri- bution. As seen in Section 9.5, use of the compound Poisson allows calculation of the total claims distribution by using a very simple recursive procedure or by using the FFT.

To proceed, note that the indicator random variable has pgf () = 1 + , and thus (9.42) may be expressed as

Y =1

Note that has a binomial distribution with parameters = 1 and = . To obtain the compound Poisson approximation, assume that has a Poisson distribution with mean . If = , then the Poisson mean is the same as the binomial mean, which should provide a good approximation if is close to zero. An alternative to equating the mean is to equate the probability of no loss. For the binomial distribution, that probability is 1 and for the Poisson distribution, it is exp(). Equating these two probabilities gives the alternative approximation = ln(1 ) . This second approximation is appropriate in the context of a group life insurance contract where a life is “replaced” upon death, leaving the Poisson intensity uPnchanged by the death. Naturally the expected number of losses is greater than =1 . An alternative choice is proposed by Kornya [98]. It uses = (1 ), and results in an expected number of losses that exceeds that using the method that equates the no-loss probabilities (see Exercise 9.71).

Regardless of the approximation used, Theorem 9.7 yields, from (9.45) using () = exp[ ( 1)],

where

exp{ [ () 1]} = exp{[ () 1]}

= () =

Y =1

() =

[ ()] (9.45)

() =

andso haspforpdf 1 X

which is a weighted average of the individual severity densities. If Pr( = ) = 1 as in life insurance, then (9.46) becomes

() = Pr( = ) = 1 X {: =}

The numerator sums all probabilities associated with amount .

X =1

,and 1 X

() =1

() = () =1

(9.46)

(9.47)

THE INDIVIDUAL RISK MODEL 269

   

   

270

AGGREGATE LOSS MODELS

0

1

2

3

4

5

6

7

8

9

10

11

12

13

14

15

16

17

18

19

()

* 60  0.9990974
* 61  0.9990986
* 62  0.9990994
* 63  0.9990995
* 64  0.9990995
* 65  0.9990996
* 66  0.9990997
* 67  0.9990997
* 68  0.9990998
* 69  0.9991022
* 70  0.9991091
* 71  0.9991156
* 72  0.9991179
* 73  0.9991341
* 74  0.9991470
* 75  0.9991839
* 76  0.9992135
* 77  0.9992239
* 78  0.9992973
* 79  0.9993307

Table 9.22

()

0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9530099 0.9534864 0.9549064 0.9562597 0.9567362 0.9601003 0.9606149

EXAMPLE 9.21

Aggregate distribution for Example 9.21.

|  |  |
| --- | --- |
| () | () |
| * 20  0.9618348 * 21  0.9618348 * 22  0.9618348 * 23  0.9618348 * 24  0.9664473 * 25  0.9664473 * 26  0.9702022 * 27  0.9702022 * 28  0.9713650 * 29  0.9713657 * 30  0.9723490 * 31  0.9735235 * 32  0.9735268 * 33  0.9735328 * 34  0.9735391 * 35  0.9735433 * 36  0.9735512 * 37  0.9735536 * 38  0.9735604 * 39  0.9735679 | * 40  0.9735771 * 41  0.9735850 * 42  0.9736072 * 43  0.9736133 * 44  0.9736346 * 45  0.9736393 * 46  0.9736513 * 47  0.9736541 * 48  0.9736708 * 49  0.9736755 * 50  0.9736956 * 51  0.9736971 * 52  0.9737101 * 53  0.9737102 * 54  0.9737195 * 55  0.9782901 * 56  0.9782947 * 57  0.9782994 * 58  0.9783006 * 59  0.9783021 |



(Example 9.20 continued) Consider the group life case of Example 9.20. De- rive a compound Poisson approximation with the means matched.

Using the comPpound Poisson approximation of this section with Poisson parameter = = 004813, the distribution function given in Table 9.22 is obtained. When these values are compared to the exact solution (not presented here), the maximum error of 0.0002708 occurs at = 0. ¤

EXAMPLE 9.22

(Example 9.19 continued) Develop compound Poisson approximations using all three methods suggested here. Compute the mean and variance for each approximation and compare it to the exact value.

Using the method that matches the mean, we have = 50(001)+25(001) = 075. The severity distribution is



(50,000) =

(75,000) = (100,000) = (150,000) =

50(001)(07) = 04667 075

25(001)(07) = 02333 075

50(001)(03) = 02000 075

25(001)(03) = 01000. 075

       

   

The mean is E() = 075(75,83333) = 56,875, which matches the exact value, and the variance is E(2) = 075(6,729,166,667) = 5,046,875,000, which exceeds the exact value.

For the method that preserves the probability of no losses, = 75 ln(099) = 0753775. For this method, the severity distribution turns out to be exactly the same as before (because all individuals have the same value of ). Thus the mean is 57,161 and the variance is 5,072,278,876, both of which exceed the previous approximate values.

Using Kornya’s method, = 75(001)099 = 0757576 and again the severity distribution is unchanged. The mean is 57,449 and the variance is 5,097,853,535, which are the largest values of all. ¤

9.11.4 Exercises

9.70 Derive (9.43) and (9.44).

9.71 Demonstrate that the compound Poisson model given by = and (9.47) produces a model with the same mean as the exact distribution but with a larger variance. Then show that the one using = ln(1 ) must produce a larger mean and even larger variance, and, nally, show that the one using = (1) must produce the largest mean and variance of all.

9.72 (\*) Individual members of an insured group have independent claims. Aggre- gate payment amounts for males have mean 2 and variance 4, while females have mean 4 and variance 10. The premium for a group with future claims is the mean of plus 2 times the standard deviation of . If the genders of the members of a group of members are not known, the number of males is assumed to have a binomial distribution with parameters and = 04. Let be the premium for a group of 100 for which the genders of the members are not known and let be the premium for a group of 40 males and 60 females. Determine .

9.73 (\*) An insurance company assumes the claim probability for smokers is 0.02 while for nonsmokers it is 0.01. A group of mutually independent lives has coverage of 1,000 per life. The company assumes that 20% of the lives are smokers. Based on this assumption, the premium is set equal to 110% of expected claims. If 30% of the lives are smokers, the probability that claims will exceed the premium is less than 0.20. Using the normal approximation, determine the minimum number of lives that must be in the group.

9.74 (\*) Based on the individual risk model with independent claims, the cumula- tive distribution function of aggregate claims for a portfolio of life insurance policies is as in Table 9.23. One policy with face amount 100 and probability of claim 0.20 is increased in face amount to 200. Determine the probability that aggregate claims for the revised portfolio will not exceed 500.

9.75 (\*) A group life insurance contract covering independent lives is rated in the three age groupings as given in Table 9.24. The insurer prices the contract so that the probability that claims will exceed the premium is 0.05. Using the normal approximation, determine the premium that the insurer will charge.

THE INDIVIDUAL RISK MODEL 271

   

   

272

AGGREGATE LOSS MODELS

Table 9.23

0

100

200

300

400

500

600

700

Distribution for Exercise 9.74.

()

0.40 0.58 0.64 0.69 0.70 0.78 0.96 1.00

  

Table 9.24

Data for Exercise 9.75.

Probability of claim per life

0.03 0.07 0.10

Data for Exercise 9.76.



Age group

Number in age group

Mean of the exponential distribution

of claim amounts

5 3 2



18—35 400 36—50 300 51—65 200

Table 9.25

Probability Service of claim

Oce visits 0.7 Surgery 0.2 Other services 0.5

Distribution of annual charges given that a claim occcurs

  

Mean

160 600 240

Variance

4,900 20,000 8,100

 

9.76 (\*) The probability model for the distribution of annual claims per member in a health plan is shown in Table 9.25. Independence of costs and occurrences among services and members is assumed. Using the normal approximation, determine the minimum number of members that a plan must have such that the probability that actual charges will exceed 115% of the expected charges is less than 0.10.

9.77 (\*) An insurer has a portfolio of independent risks as given in Table 9.26. The insurer sets and such that aggregate claims have expected value 100,000 and minimum variance. Determine .

9.78 (\*) An insurance company has a portfolio of independent one-year term life policies as given in Table 9.27. The actuary approximates the distribution of claims in the individual model using the compound Poisson model in which the expected number of claims is the same as in the individual model. Determine the maximum value of such that the variance of the compound Poisson approximation is less than 4,500.

   

   

9.79 (\*) An insurance company sold one-year term life insurance on a group of 2,300 independent lives as given in Table 9.28. The insurance company reinsures amounts in excess of 100,000 on each life. The reinsurer wishes to charge a premium that is sucient to guarantee that it will lose money 5% of the time on such groups. Obtain the appropriate premium by each of the following ways:

(a) Using a normal approximation to the aggregate claims distribution. (b) Using a lognormal approximation.

(c) Using a gamma approximation. (d) Using the compound Poisson approximation that matches the means.

9.80 A group insurance contract covers 1,000 employees. An employee can have at most one claim per year. For 500 employees, there is a 0.02 probability of a claim and when there is a claim, the amount has an exponential distribution with mean 500. For 250 other employees, there is a 0.03 probability of a claim and amounts are exponential with mean 750. For the remaining 250 employees, the probability is 0.04 and the mean is 1,000. Determine the exact mean and variance of total claims payments. Next, construct a compound Poisson model with the same mean and determine the variance of this model.

9.12 TVaR FOR AGGREGATE LOSSES

The calculation of the TVaR for continuous and discrete distributions is discussed in Sections 5.5 and 6.14. So far in the current chapter, we have dealt with the calculation of the exact (or approximating) distribution of the sum of a random number of losses. Clearly, the shape of this distribution depends on the shape of both the discrete frequency distribution and the continuous (or possibly discrete) severity distribution. On one hand, if the severity distribution is light-tailed and the frequency distribution is not, then the tail of the aggregate loss distribution will be largely determined by the frequency distribution. Indeed, in the extreme case where all losses are of equal size, the shape of the aggregate loss distribution

TVaR FOR AGGREGATE LOSSES 273



Class

Standard Substandard

Class

Table 9.26 Data for Exercise 9.77.

Probability of claim Benet

Number of risks

3,500 2,000

Probability of a claim

 

Table 9.27

Number in class

0.2 0.6

Data for Exercise 9.78.

Benet amount

 

1 500 0.01 2 500 2 0.02

    

   

274 AGGREGATE LOSS MODELS Table 9.28

Data for Exercise 9.79.

Probability of death

0.10 0.02 0.02 0.10 0.10



Class Benet

1 2 3 4 5

amount

100,000 200,000 300,000 200,000 200,000

Number of policies

500

500

500

300

500

 

is completely determined by the frequency distribution. severity distribution is heavy-tailed and the frequency is tail of the aggregate loss distribution will be determined by the shape of the severity distribution because extreme outcomes will be determined with high probability by a single, or at least very few, large losses. In practice, if both the frequency and severity distribution are specied, it is easy to compute the TVaR at a specied quantile.

9.12.1 TVaR for discrete aggregate loss distributions

We discuss in earlier sections in this chapter the numerical evaluation of the aggre- gate loss distribution requires a discretization of the severity distribution resulting in a discretized aggregate loss distribution. We, therefore, give formulas for the dis- crete case. Consider the random variable representing the aggregate losses. The overall mean is the product of the means of the frequency and severity distributions. Then the TVaR at quantile for this distribution is5

T V a R ( ) = E ( | P ) =+ ()+()

(9.48)

On the other hand, if the not, then the shape of the



1() ()+()= ()()+ ( )+()

Noting that

XXXX

= E ( ) + X( ) + ( )

=E() + we see that, because 0, the last sum in equation (9.49) is taken over a nite

( )() number of points, the points of support up to the quantile

5The quantile must be one of the points of support of the aggregate loss distributions. If the selected quantile is not such a point, the TVaR can be calculated at the two adjacent points and the results interpolated to get an approximate value of the desired TVaR.

(9.49)

   

   

Then the result of the equation (9.49) can be substituted into equation (9.48) to obtain the value of the TVaR. The value of the TVaR at high quantiles depends on the shape of the aggregate loss distribution. For certain distributions, we have ana- lytic results that can give us very good estimates of the TVaR. To get those results, we rst need to analyze the extreme tail behavior of the aggregate loss distribution. We rst focus on frequency distributions and then on severity distributions.

9.12.2 Aggregate TVaR for some frequency distributions

We use the notation () (), to denote that lim () = 1

() Denition 9.14 A function () is said to be slowly varying at innity if

()(), for all 0

The logarithm function ln() and any constant function are slowly varying at innity while the exponential function exp() is not.

We now consider frequency distributions that satisfy () (9.50)

where 0 1 and () is slowly varying at innity. Distributions satisfying formula (9.50) include the negative binomial (see Exercise 9.81), the geometric, the logarithmic, Poisson—ETNB (when 1 0) (see Teugels and Willmot [172]) including the Poisson—inverse Gaussian, and mixed Poisson distributions with mixing distributions that are suciently heavy-tailed (see Willmot [186]) and many compound distributions (see Willmot [185]).

We also consider severity distributions that have a moment generating function. In addition, we assume that there exists a number 0 satisfying

() = 1 (9.51)

In very general terms, this condition ensures that the severity distribution is not too heavy-tailed. For distributions whose moment generating functions increase indenitely, the condition is always satised. However, some distributions (e.g., inverse Gaussian) have moment generating functions that have an upper limit, in which case condition (9.51) is satised only for some values of .

The following theorem of Embrechts, Maejima, and Teugels [40] gives the as- ymptotic shape of the tail of the aggregate loss distribution for large quantiles.

Theorem 9.15 Let denote that probability function of a counting distribution satisfying condition (9.50), and let () denote the mgf of a nonarithmetic sever- ity distribution satisfying condition (9.51). Then if 0() , the tail of the corresponding aggregate loss distribution satises

1 () () (9.52) [0()]+1

TVaR FOR AGGREGATE LOSSES 275

      

   

276 AGGREGATE LOSS MODELS

This theorem shows that the tail of the aggregate loss distribution looks like the product of a gamma density and a slowly varying function. The terms in the denominator form the necessary normalizing constant. The asymptotic formula for the tail in Theorem 9.15 can be used as an approximation for the tail for high quantiles. Having obtained this, we can obtain approximate values of the TVaR from

TVaR () = E( |R ) ()()

=+  R [1 ()]



1() =+



1() In this situation we can get an asymptotic formula for the TVaR. Formula (9.52)

may be expressed as

where

 and () varies slowly at innity because () does. From Grandell [57, p. 181], it

follows from (9.53) that

Z[1()]()

Therefore, R[1 ()] 1 1()

and we obtain the TVaR approximately as TVaR () = E( | )

+ 1

which is exactly the TVaR for the exponential distribution with mean 1. In this case, the extreme tail becomes approximately exponential, and so the conditional expected excess over the quantile is constant.

9.12.3 Aggregate TVaR for some severity distributions

In this subsection, we consider the aggregate TVaR based on properties of the severity distribution rather than the frequency distribution. Explicit expressions are normally not available (the case with mixed Erlang severity distributions is a notable exception, as follows from Exercise 9.85), but using dierent arguments than those used in Section 9.12.2, we can still obtain asymptotic results for the tail and the TVaR of the aggregate loss distribution.

1 () () (9.53)

() = () [0()]+1

        

   

TVaR FOR AGGREGATE LOSSES 277 We consider the class of severity distributions satisfying, for some 0, the

conditions

and

lim 12() =2() 1()

lim 1()= 0 1()

(9.54)

(9.55)

 

When = 0, the subclass is referred to as the class of subexponential distributions. They are heavy tailed and can be shown to have no moment generating function so that Theorem 9.15 cannot be used. The subexponential class is broad and includes many of the distributions discussed in Chapter 5. A notable subclass of the class of subexponential distributions is the class of distribution with regularly varying tails, that is, those which satisfy

() () (9.56)

where () is slowly varying at innity and 0  The transformed beta family of distributions satises (9.56) with in (9.56)

replaced by and () constant (see Exercise 9.82). If 0, distributions satisfying (9.54) and (9.55) are sometimes called medium-

tailed. This includes distributions with pdf satisfying () 1 (9.57)

with 0. The inverse Gaussian distribution satises (9.57) with = 05 and = (22) (see Exercise 9.83).

Teugels [171] shows that if (9.54) and (9.55) hold, then 1 () 0[()]() (9.58)

as long as () for some (). In the subexponential case, (0) = 1 and 0(1) =E[]

The class of medium-tailed distributions may or may not determine the tail of the aggregate loss distribution. As an illustration, suppose the claim frequency distribution satises (9.50) and the severity distribution is medium-tailed. If () is the claim frequency pgf, then by the ratio test for convergence, its radius of convergence is 1, that is, |()| if || 1 and |()| = if || 1. Note that the niteness of (1) is not specied. Therefore, if () 1, then 0[()] and the preceding medium-tailed result applies. If () 1, however, 0 satisfying (9.51) may be found, and Theorem 9.15 applies.

The asymptotic formula (9.58) allows for asymptotic estimates of the TVaR.

EXAMPLE 9.23

Derive an asymptotic formula for the aggregate TVaR in the case where the severity distribution has regularly varying tails given by (9.56).

If (9.56) holds, then using Grandell [57, p. 181], if 1,

Z 1

[1()]()1

     

   

278 AGGREGATE LOSS MODELS

Therefore R[1 ()]

 

and with =

1 () 1 lim TVaR() =1+ lim TVaR() =1+ 1

1

  

That is, if 1,

TVaR() 1



Now, suppose that the claim severity distribution has pdf () and hazard rate function () = ()() that satises

() ()  where () is a (simpler) limiting hazard rate function. Then, by L’Hôpital’s

rule,

lim

R[1 ()] 1 ()

= lim R[1 ()] 0[()]()

= lim 1() 0[()]()

= lim () ()

=1 ()

    

Thus, with replaced by , TVaR()

1 ()

(9.59)

(9.60)

¤



yielding the approximate formula for large , TVaR() + 1

 In any particular application, it remains to identify ().

EXAMPLE 9.24



Approximate the aggregate TVaR if the claim severity pdf satises (9.57) with 0.

By L’Hôpital’s rule,

1 lim = lim

( 1)2 1

()

  

() = lim

1

()

μ()

¶

 

=

1

    

   

TVaR FOR AGGREGATE LOSSES 279 Thus, () ()1 , and consequently

() 1 () = () ()1 =

This result means that () = , yielding the approximation TVaR() + 1

for large values of . EXAMPLE 9.25

Approximate the aggregate TVaR in the case with a lognormal severity dis- tribution.

Embrechts, Goldie, and Veraverbeke [38] show that the lognormal distrib- ution is subexponential. The pdf is

()=1 exp 1 (ln)2 ̧0 2 22

To identify (), consider the function

  

¤

   

exp£ 1 (ln)2¤ μ ¶ ()= 22 = 2 ()

   

Note that

ln ln 0() = 1(ln)2 1 ̧exp 1 (ln)2 ̧

  

2 ̧ 22 = 2 1 +(ln)2 ()

 

2 Because () 0 as , L’Hôpital’s rule yields

lim () = lim ()

 

0() = 1 lim1+(ln)2 ̧1

= 2 1μ¶1 2 ̧

()

 

2 2

  

Thus,

() 2()= 2 ln exp 22(ln)

     

In turn,

()= () 2() = ln () () 2

        

   

280 AGGREGATE LOSS MODELS That is, the lognormal distribution has asymptotic hazard rate

()= ln 2

Finally, the asymptotic approximation for the TVaR of the lognormal distri-



bution is

2 T Va R ( ) = ( | ) + l n



which increases at a rate faster than linear. ¤ 9.12.4 Summary

Section 9.12 and related results suggest that the tail behavior of the aggregate loss distribution is essentially determined by the heavier of the frequency and severity distributions. In particular,

1. If the frequency distribution is suciently heavy-tailed and the severity distri- bution is light-tailed, the tail of the aggregate loss distribution is determined by the frequency distribution through Theorem 9.15.
2. If the severity distribution is suciently heavy-tailed and if the frequency distribution has a moment generating function, and is thus light-tailed, the tail of the aggregate loss distribution looks like a rescaled severity distribution.
3. For medium-tailed severity distributions, such as the inverse Gaussian, the tail of the aggregate loss distribution may or may not be determined by the severity distribution, depending on the parameter values of that severity dis- tribution.

9.12.5 Exercises

9.81 Use Stirling’s formula, () 205

to show that the negative binomial pf (Appendix B) satises

(1+) 1μ ¶ () 1 +

  

9.82 Prove that the transformed beta tail (Appendix A) satises ( + ) 3 ́

()(+1)() 9.83 Prove that the inverse Gaussian pdf (Appendix A) satises

 

() r 15(22) 2

     

   

TVaR FOR AGGREGATE LOSSES 281 9.84 Let be an arbitrary claim severity random variable. Prove that (9.59)

implies that

E(|)E(|)  which means that the aggregate mean excess loss function is the same asymptoti-

cally as that of the claim severity.

9.85 Consider a compound distribution with mixed Erlang severities from Example 9.9. Prove, using Exercise 9.41 that

P 3P ̄ ́() =0 = !

TVaR()=+ P ̄() =0 !

where () = 3P=0 ́ = P=0 , and ̄ = P=+1 = 1P=0

   